Control Systems & Transfer Functions

Transfer functions may be defined for a linear, stationary (constant parameter system) system.

For continuous time input $x(t)$ and output $y(t)$, the transfer function $H(s)$ is :

$$H(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{x(t)\}} = \frac{Y(s)}{X(s)}$$

$$\therefore \quad Y(s) = H(s)\,X(s)$$

where $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$ ← Laplace Transform

E.g. Harmonic oscillator :

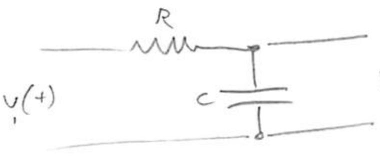
$$m\ddot{y}(t) + \Gamma \dot{y}(t) + k y(t) = r(t)$$

$$\rightarrow \quad m s^2 Y(s) + \Gamma s Y(s) + k Y(s) = R(s)$$

Transfer function

$$\hookrightarrow \quad G(s) = \frac{\text{Output}}{\text{Input}} = \frac{Y(s)}{R(s)} = \frac{1}{m s^2 + \Gamma s + k}$$

E.g. RC network :



$$V_1(s) = \left(R + \frac{1}{Cs}\right) I(s)$$

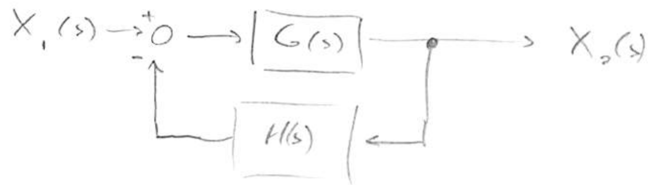
$$V_2(s) = I(s)\,\frac{1}{Cs}$$

$$\therefore \quad V_2(s) = \frac{\frac{1}{Cs} V_1(s)}{R + \frac{1}{Cs}}$$

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1}$$

Block Diagram

Feedback & Added Complexity



$$X_2(s) = G(s)\left[X_1(s) - H(s)\,X_2(s)\right]$$

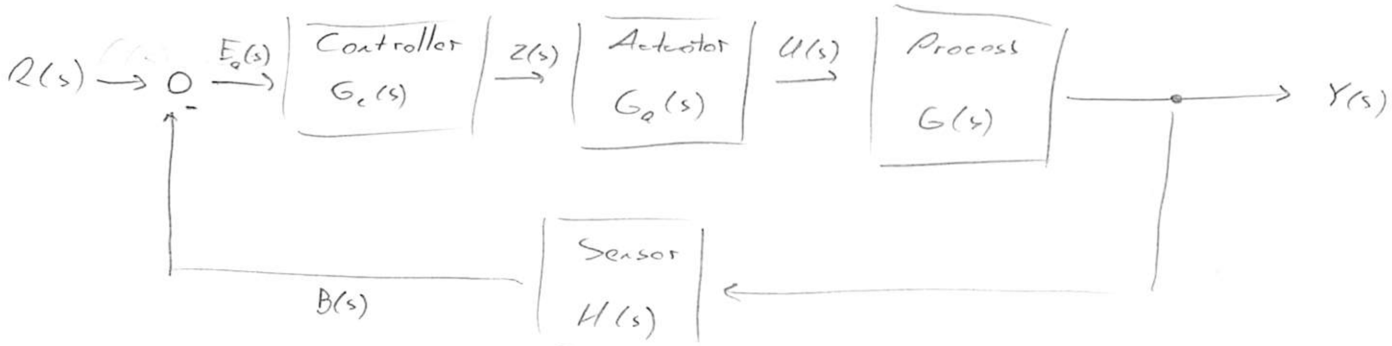
$$X_2(s) = G(s)\,X_1(s) - G(s)\,H(s)\,X_2(s)$$

$$\left[1 + G(s)\,H(s)\right] X_2(s) = G(s)\,X_1(s)$$

$$\frac{X_2(s)}{X_1(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$ ← Transfer Function

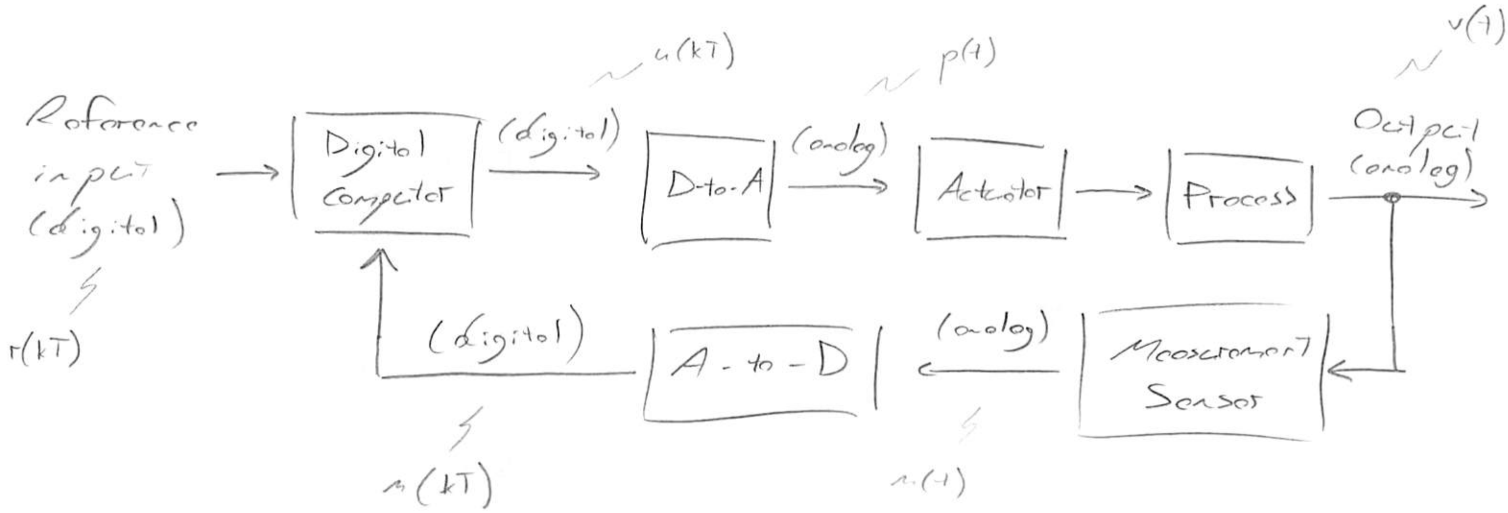
$$X_1(s) \longrightarrow \boxed{\frac{G(s)}{1 + G(s)H(s)}} \longrightarrow X_2(s)$$

E.g.

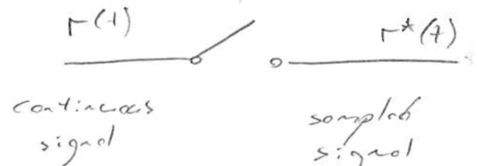


$$\frac{\text{Output}}{\text{Input}} = \frac{Y(s)}{R(s)} = \frac{G(s)\,G_a(s)\,G_c(s)}{1 + G(s)\,G_a(s)\,G_c(s)\,H(s)}$$ ← closed loop transfer function

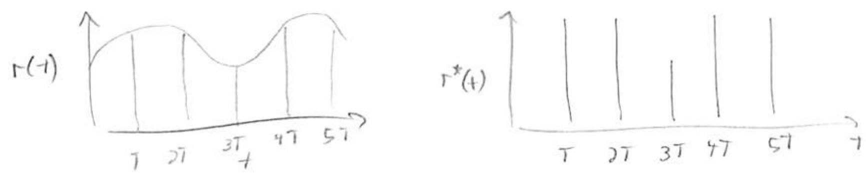
Digital Control Systems

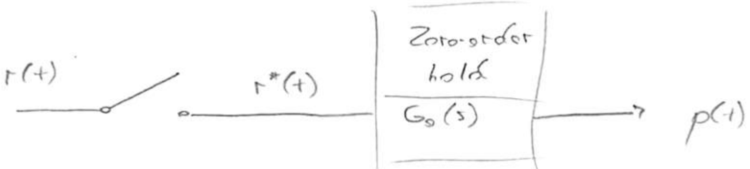


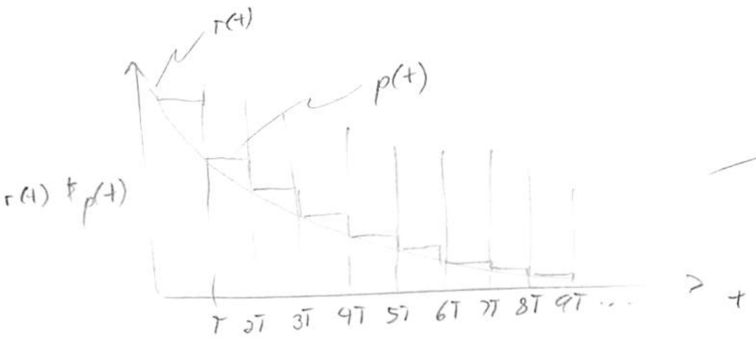
Ideal Sampler:



$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\, \delta\left(\frac{t - nT}{T}\right)$$







$$p(t) = \sum_{n=-\infty}^{\infty} r(nT) \cdot \mathrm{rect}\left(\frac{t - T/2 - nT}{T}\right)$$

w/

$$\mathrm{rect}(t) = \begin{cases} 0 & |t| > \frac{1}{2} \\ \frac{1}{2} & |t| = \frac{1}{2} \\ 1 & |t| < \frac{1}{2} \end{cases}$$

$$\lim_{T \to 0} p(t) = r(t)$$

Zero-order hold transfer function:

Laplace transform

$$F(s) = \int_0^\infty f(t) e^{-st} dt = \mathcal{L}(f(t))$$

$$G_0(s) = \mathcal{L}\left(\mathrm{rect}\left(\frac{t - \frac{T}{2}}{T}\right)\right)$$

$$= \int_0^\infty \mathrm{rect}\left(\frac{t - \frac{T}{2}}{T}\right) e^{-st} dt$$

$$= \int_0^T e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^T = \frac{1}{s} - \frac{1}{s} e^{-sT}$$

$$G_0(s) = \frac{1 - e^{-sT}}{s}$$

Z - Transform → for discrete time systems

$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\, \delta\left(\frac{t - nT}{T}\right)$$

$$\mathcal{L}\{r^*(t)\} = \sum_{n=0}^{\infty} r(nT)\, e^{-nsT}$$

infinite series

We define:

$$z = e^{sT} \Leftarrow$$

$$Z\{r(t)\} = R(z) = \sum_{n=0}^{\infty} r(nT)\, z^{-n}$$

E.g. step function $u(t)$ → $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

$$Z\{u(t)\} = U(z) = \sum_{n=0}^{\infty} u(nT)\, z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Recall

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$

E.g. Exponential

$$f(t) = e^{-at}$$

$$Z\{f(t)\} = F(z) = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty} \left(z e^{aT}\right)^{-n}$$

$$\therefore \quad F(z) = \frac{1}{1 - (z e^{aT})^{-1}} = \frac{z}{z - e^{aT}}$$

E.g. Sinusoid

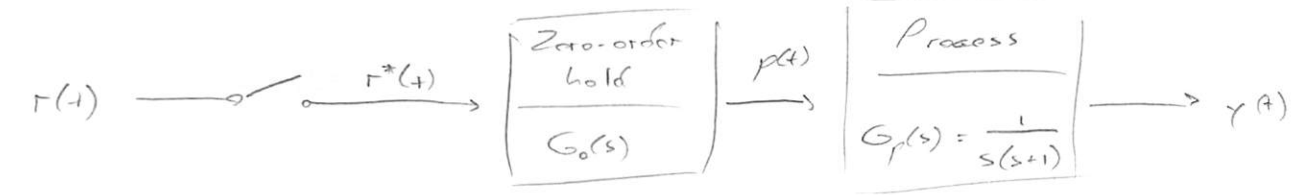
$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \frac{e^{i\omega t}}{2i} - \frac{e^{-i\omega t}}{2i}$$

$$F(z) = \frac{1}{2i}\left(\frac{z}{z - e^{i\omega T}} - \frac{z}{z - e^{-i\omega T}}\right)$$

$$F(z) = \frac{1}{2i}\left(\frac{z\left(e^{i\omega T} - e^{-i\omega T}\right)}{z^2 - z\left(e^{i\omega T} + e^{-i\omega T}\right) + 1}\right)$$

$$= \frac{z \sin(\omega T)}{z^2 - 2z\sin(\omega T) + 1}$$

Transfer Function of Open Loop System



$$G_0(s) = \frac{1 - e^{-sT}}{s}$$

Transfer function :

$$\frac{Y(s)}{R^*(s)} = G_0(s)\,G_p(s) = G(s) = \frac{1 - e^{-sT}}{s^2(s+1)}$$

$$G(s) = \left(1 - e^{-sT}\right)\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

Using table .. (slide) ⟶

$$G(z) = Z\{G(s)\} = \left(1 - z^{-1}\right)\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$G(z) = \frac{\left(ze^{-T} - z + Tz\right) + \left(1 - e^{-T} - Te^{-T}\right)}{(z-1)\,(z-e^{-T})}$$

Let $T = 1$ ...

$$G(z) = \frac{ze^{-1} + 1 - 2e^{-1}}{(z-1)\,(z-e^{-1})} = \frac{0.3678z + 0.2644}{(z-1)(z-0.3678)} = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678}$$

Response of the system to a unit impulse $R(z) = 1$

↳ $$Y(z) = G(z)\cdot 1 = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678}$$

$$Y(z) = 0.3678z^{-1} + 0.7675z^{-2} + 0.9145z^{-3} + \dots$$

Recall the definition of a z-Transform:

$$Z\{z(t)\} = Y(z) = \sum_{n=0}^{\infty} y(nT)\, z^{-n}$$

$\therefore$ we have calculated the $y(nT)$'s, i.e. the samples.

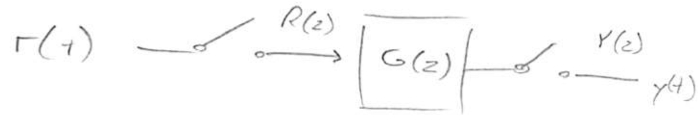
$y(0) = 0$

$y(T) = 0.3678$

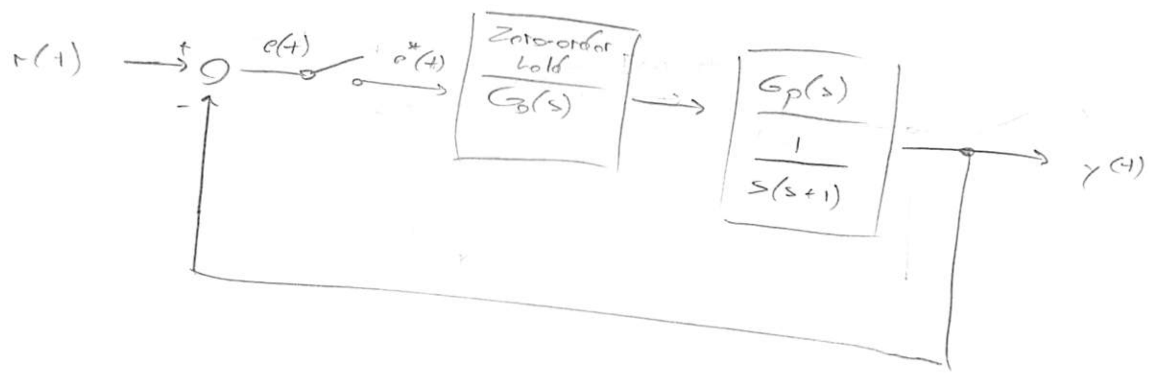
$y(2T) = 0.7675$

$y(3T) = 0.9145$

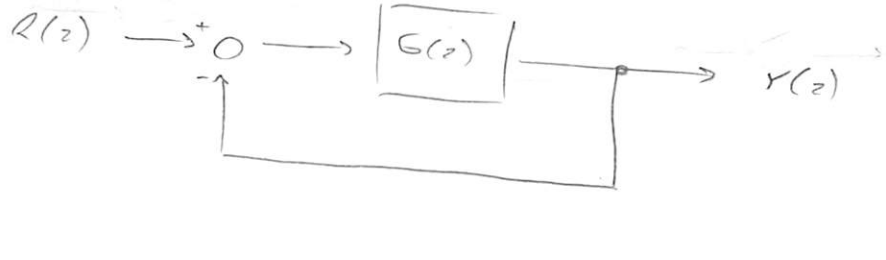
$\vdots$



Closed-Loop Feedback Sampled Systems



$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$



Substituting our $G(z) = \dfrac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678}$ into the previous equation ...

$$\frac{Y(z)}{R(z)} = \frac{0.3678z + 0.2644}{z^2 - z + 0.6322}$$

The input is a unit step,

$\therefore \quad R(z) = \dfrac{z}{z-1}$

It follows that :

$$Y(z) = \frac{z\,(0.3678z + 0.2644)}{(z-1)(z^2 - z + 0.6322)}$$

$$Y(z) = 0.3678\,z^{-1} + z^{-2} + 1.4\,z^{-3} + 1.4\,z^{-4} + 1.147\,z^{-5} + \ldots$$

PID Controller

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{d}{dt} e(t)$$

Laplace transform :

$$G(s) = K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}$$

z transform : (see table ... slide)

$$G(z) = K_p + \frac{K_i T z}{z - 1} + \frac{K_d (z - 1)}{T z}$$