PID Controller

$$u(t) = K_p\, e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{d}{dt} e(t)$$

proportional term: output $\propto$ to error

integral term: output $\propto$ magnitude & duration of error

derivative term: output $\propto$ slope of error

Laplace Transform:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$$

Express the PID equation in terms of inverse Laplace transforms & divide out the integrals...

$$U(s) = K_p E(s) + K_i \frac{E(s)}{s} + K_d s E(s)$$

Transfer function: $G(s) = \dfrac{U(s)}{E(s)} = K_p + \dfrac{K_i}{s} + K_d s$

$$G(s) = \frac{K_d s^2 + K_p s + K_i}{s}$$

Recall:

$$\int u\,dv = uv - \int v\,du$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= f(t)\left(\frac{e^{-st}}{-s}\right)\Big|_0^\infty - \int_0^\infty \left(\frac{e^{-st}}{-s}\right) f'(t)\,dt$$

$$= \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}\{f'(t)\}$$

$$\therefore \mathcal{L}\{f'(t)\} = s \cdot \mathcal{L}\{f(t)\} - f(0)$$

$$\hookrightarrow \boxed{\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)}$$

$\vdots$

Conso... slow ...

$$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

$$\hookrightarrow \boxed{\mathcal{L}\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)}$$

Heaviside func:

$$h(t) = \int_{-\infty}^{t} \delta(s)\,ds$$

or

$$h(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \mathcal{L}\left\{\int_{-\infty}^{\infty} h(\tau)\, f(t-\tau)\,d\tau\right\}$$

$$= \mathcal{L}\{h(t) * f(t)\}$$

Transfor of Convolution is producto of Transforms

$$= \mathcal{L}\{h(t)\}\,\mathcal{L}\{f(t)\} = H(s)\,F(s)$$

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{1}{s}F(s)}$$

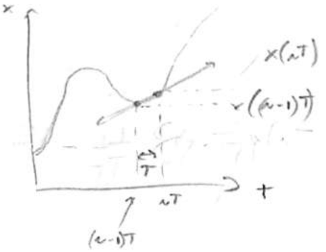
Digital implementation

Convolution

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$$G(s) = K_p + \frac{K_i}{s} + K_d\, s$$

For derivative:



$$u(nT) = \left.\frac{dx}{dt}\right|_{t=nT} = \frac{1}{T}\Big(x(nT) - x([n-1]T) \Big)$$

$$u(nT) = \frac{1}{T}\,\delta(t) * x(t) - \frac{1}{T}\,\delta(t-T) * x(t)$$

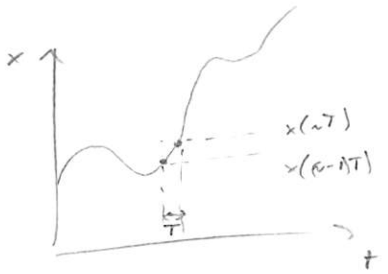
z- Transform:

$$U(z) = \frac{1}{T}\, 1 \cdot X(z) - \frac{1}{T}\, z^{-1} \cdot X(z)$$

$$U(z) = \frac{1 - z^{-1}}{T}\, X(z) = \frac{z-1}{zT}\, X(z)$$

$$\frac{U(z)}{X(z)} = \frac{z-1}{zT} \quad \Longleftarrow$$

For integration:



$$u(nT) = u((n-1)T) + T\, x(nT)$$

$$u(nT) = \delta(t-T) * u(t) + T\,\delta(t) * x(t)$$

z- Transform:

$$U(z) = z^{-1} \cdot U(z) + T \cdot X(z)$$

$$\frac{U(z)}{X(z)} = \frac{T}{1 - z^{-1}} = \frac{Tz}{z-1} \quad \Longleftarrow$$

Therefore the full z-Transform transfer function for a PID is:

$$G(z) = K_p + K_i \frac{Tz}{z-1} + K_d \frac{z-1}{zT}$$

compared to the Laplace transform:

$$G(s) = K_p + \frac{K_i}{s} + K_d s$$

z-Transform to difference equation:

$$U(z) = G(z)\,E(z)$$

$$U(z) = \left[K_p + K_i \frac{T}{1-z^{-1}} + K_d \frac{1-z^{-1}}{T}\right] E(z)$$

$$U(z) = \frac{\overbrace{\left(K_p + K_i T + \frac{K_d}{T}\right)}^{K_1} + \overbrace{\left(-K_p - 2\frac{K_d}{T}\right)}^{K_2} z^{-1} + \overbrace{\frac{K_d}{T}}^{K_3} z^{-2}}{1 - z^{-1}} \quad E(z)$$

$$U(z) - z^{-1}U(z) = \left[K_1 + K_2 z^{-1} + K_3 z^{-2}\right] E(z)$$

Since z-transform of $\delta(t - nT)$ is z^{-n},

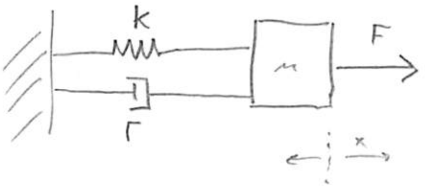
$\therefore$

$$u(t) - \delta(t-T) * u(t) = K_1 e(t) + K_2\, \delta(t-T) * e(t) + K_3\, \delta(t-2T) * e(t)$$

$$u(nT) - u([n-1]T) = K_1 e(nT) + K_2 e([n-1]T) + K_3 e([n-2]T)$$

$\rightarrow$ Difference equation for digital implementation in a program or code as a function of sample times.

Example Problem



$$m\ddot{x} + \Gamma \dot{x} + kx = F$$

initial cond.

$x(0) = 0$

$\dot{x}(0) = 0$

Laplace transform :

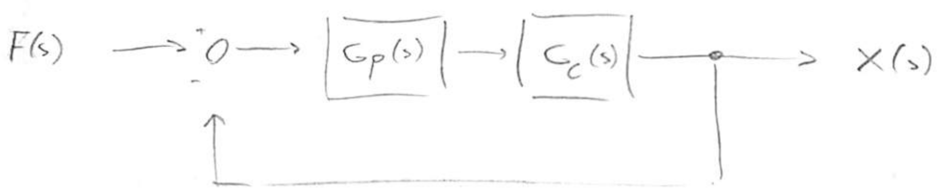
$$ms^2 X(s) + \Gamma s X(s) + k X(s) = F(s)$$

Transfer Function :

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + \Gamma s + k} = G_p(s)$$

→ Design controller :

Prop. only



$$\frac{X(s)}{F(s)} = \frac{G_p(s)\, G_c(s)}{1 + G_p(s)\, G_c(s)}$$

only prop control

$$G_c(s) = k_p$$

$$= \frac{\dfrac{k_p}{ms^2 + \Gamma s + k}}{1 + \dfrac{k_p}{ms^2 + \Gamma s + k}} = \frac{k_p}{ms^2 + \Gamma s + (k + k_p)}$$

Prop., Int., & Deriv.

$$G_c(s) = k_p + \frac{k_i}{s} + k_d s$$

$$\frac{X(s)}{F(s)} = \frac{\dfrac{k_p + \frac{k_i}{s} + k_d s}{ms^2 + \Gamma s + k}}{1 + \dfrac{k_p + \frac{k_i}{s} + k_d s}{ms^2 + \Gamma s + k}}$$

$$= \frac{k_p + \frac{k_i}{s} + k_d s}{ms^2 + \Gamma s + k + k_p + \frac{k_i}{s} + k_d s}$$

$$\frac{X(s)}{F(s)} = \frac{k_d s^2 + k_p s + k_i}{ms^3 + (\Gamma + k_d)s^2 + (k + k_p)s + k_i}$$