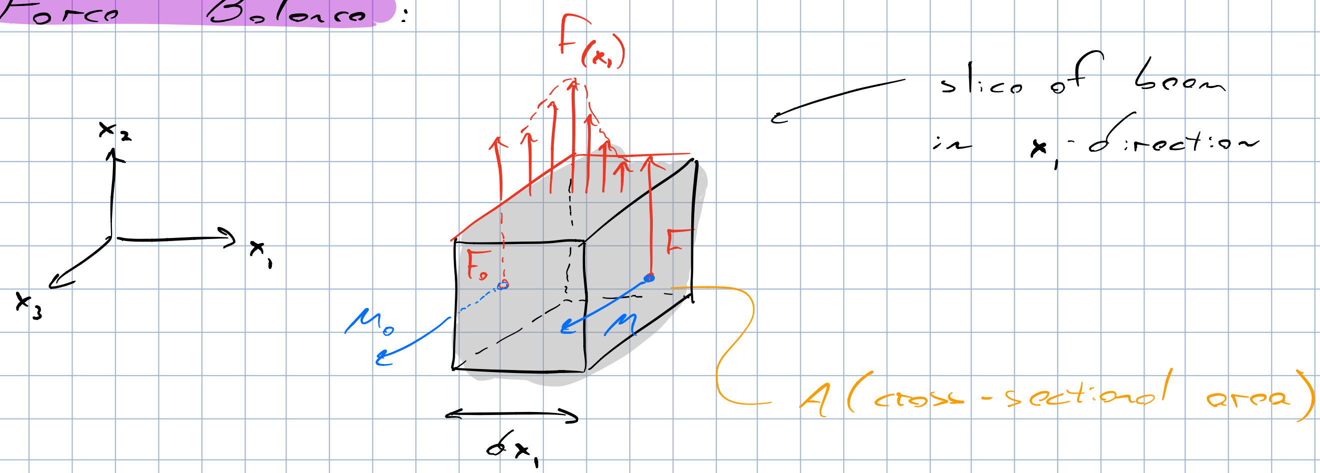
Lecture 2 (23.09.2020)

Cantilever Statics

I. Beam Bending Formula

Force Balance:



$$F_0 + \int_0^{l} f(x_1')\, A\, dx_1' + F(l) = 0 \quad \leftarrow \text{full beam}$$

$$F_0 + \int_0^{x_1} f(x_1')\, A\, dx_1' + F(x_1) = 0 \quad \leftarrow \text{for any part of the beam}$$

$$F(x_1) = -F_0 - \int_0^{x_1} f(x_1')\, A\, dx_1'$$

$$\frac{dF}{dx_1} = -f(x_1)\, A \longrightarrow \boxed{dF = -f(x_1)\, A\, dx_1}$$

Torque Balance

$$M_0 + \int_0^{l} x_1' f(x_1')\, A\, dx_1' + l F(l) + M(l) = 0 \quad \leftarrow \text{full beam}$$

$$M_0 + \int_0^{z} z' f(z')\, A\, dz' + x_1 F(x_1) + M(x_1) = 0 \quad \leftarrow \text{For any part of the beam}$$

$$M(x_1) = -M_0 - \int_0^{x_1} x_1' f(x_1') A \, dx_1' - x_1 F(x_1)$$

$$\frac{dM}{dx_1} = -x_1 f(x_1) A - F(x_1) - x_1 \frac{dF}{dx_1}(x_1)$$

$$\boxed{dM = -x_1 f(x_1) A \, dx_1 - F \, dx_1 - x_1 \, dF}$$

Together :

$$\frac{dF}{dx_1} = -f(x_1) A$$

$$\frac{dM}{dx_1} = -x_1 f(x_1) A - F - x_1 \frac{dF}{dx_1}$$

$$\frac{dM}{dx_1} = \cancel{-x_1 f(x_1) A} - F + \cancel{x_1 f(x_1) A}$$

$$\therefore \quad \frac{d^2 M}{dx_1^2} = f(x_1) A$$

Recall from before , $\boxed{\dfrac{d^2 u_2}{dx_1^2} = \dfrac{M_0}{E I_3}}$

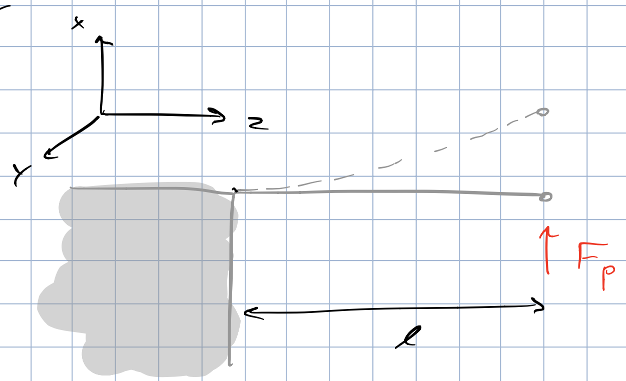
Therefore : $\dfrac{d^4 u_2}{dx_1^4} = \dfrac{1}{E I_3} \dfrac{d^2 M}{dx_1^2}$

Beam Bending Formula → $\boxed{\dfrac{d^4 u_2}{dx_1^4} = \dfrac{f(x_1) A}{E I_3}}$

Let's use these two formulas to solve some simple static examples:

Example 1:

Take a singly-clamped beam with a point force applied to its end.



Boundary conditions: (1) $u_x(0) = 0$ ← no displacement at clamp

(2) $\frac{du_x}{dz}(0) = 0$ ← no bending at clamp

(3) $\frac{d^2u_x}{dz^2}(\ell) = 0$ ← no torque applied to end

Note that applied force density is:

$$F_p \cdot \frac{\delta(z-\ell)}{A} \quad [N/m^3]$$

$$\frac{d^4u_x}{dz^4} = \frac{f(z)A}{EI_y} = \frac{1}{EI_y}\left(-F_p\,\delta(z) + F_p\,\delta(z-\ell)_-\right)$$

$$\frac{d^3u_x}{dz^3} = -\frac{F_p}{EI_y} \quad \text{for } z < \ell$$

$$\frac{d^2u_x}{dz^2} = -\frac{F_p}{EI_y}z + c_1 \quad \text{where } c_1 = \frac{F_p\ell}{EI_y}$$

From (3)

$$\frac{du_x}{dz} = -\frac{F_p}{2EI_y}z^2 + \frac{F_p\ell}{EI_y}z + c_2 \quad \text{where } c_2 = 0$$

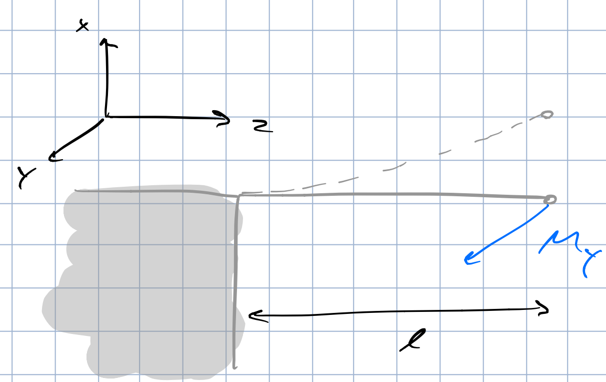
From (2)

$$u_x = -\frac{F_p}{6EI_y} z^3 + \frac{F_p \ell}{2EI_y} z^2 + c_3 \quad \text{where } c_3 = 0 \text{ from (1)}$$

$$\boxed{u_x = \frac{F_p}{2EI_y}\left(\ell z^2 - \frac{z^3}{3}\right)}$$

Example 2:

Now take a singly-clamped beam with a point torque applied to its end.



Boundary conditions:

(1) $u_x(0) = 0$ ← no displacement at clamp

(2) $\frac{du_x}{dz}(0) = 0$ ← no bending at clamp

$$\frac{d^2 u_x}{dz^2} = \frac{M_y}{EI_y}$$

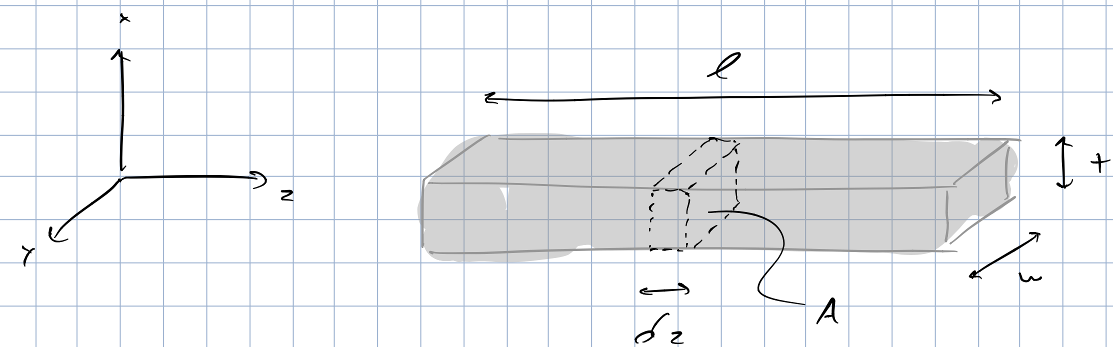
$$\frac{du_x}{dz} = \frac{M_y}{EI_y} z + c_1 \quad \text{where } c_1 = 0 \text{ from (2)}$$

$$u_x = \frac{M_y}{2EI_y} z^2 + c_2 \quad \text{where } c_2 = 0 \text{ from (1)}$$

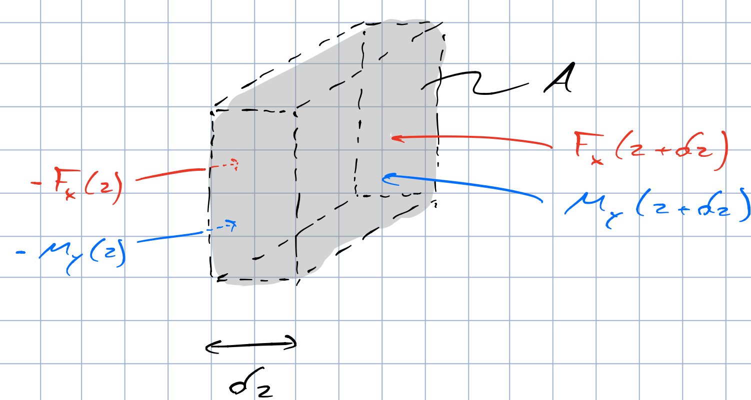
$$\boxed{u_x = \frac{M_y}{2EI_y} z^2}$$

Cantilever Dynamics

I. Flexural Vibrations



Let's calculate the dynamic behavior of this beam as the end flexes in the x-direction.



Balance Forces:

$$F_x(z+dz) - F_x(z) = \underbrace{\rho A\, dz}_{\text{mass}} \; \underbrace{\frac{d^2 u_x}{dt^2}}_{\text{acceleration}}$$

Balance Torques:

$$M_y(z+\delta z) - M_y(z) + F_x(z+\delta z)\,\delta z = 0$$

Expanding out for small δz around z:

$$\boxed{\begin{aligned} \frac{\partial F_x}{\partial z} &= \rho A \frac{\partial^2 u_x}{\partial t^2} \\ \frac{\partial M_y}{\partial z} &= -F_x(z) \end{aligned}} \quad \Bigg\} \quad \boxed{\frac{\partial^2 M_y}{\partial z^2} = -\rho A \frac{\partial^2 u_x}{\partial t^2}}$$

Recall from before: $\dfrac{d^2 u_x}{dz^2} = \dfrac{M_y}{E I_y}$

$$\therefore \quad M_y = E I_y \frac{\partial^2 u_x}{\partial z^2}$$

$$E I_y \frac{\partial^4 u_x}{\partial z^4} = -\rho A \frac{\partial^2 u_x}{\partial t^2}$$

$$\longrightarrow \quad \boxed{E I_y \frac{\partial^4 u_x}{\partial z^4} + \rho A \frac{\partial^2 u_x}{\partial t^2} = 0}$$

Let's now assume harmonic time dependence for the displacement:

$$u_x(z,t) = u_s(z)\, e^{-i\omega t}$$

$$\therefore \quad \frac{d^4 u_x}{dz^4}(z) = \left(\frac{\rho A}{E I_y}\right) \omega^2 u_x(z)$$

Define: $\beta = \left(\frac{\rho A}{E I_y}\right)^{1/4} \omega^{1/2}$

Then solutions are of the form:

$$u_x(z) = e^{kz} \quad \text{with} \quad k = \pm\beta, \pm i\beta$$

A general solution is:

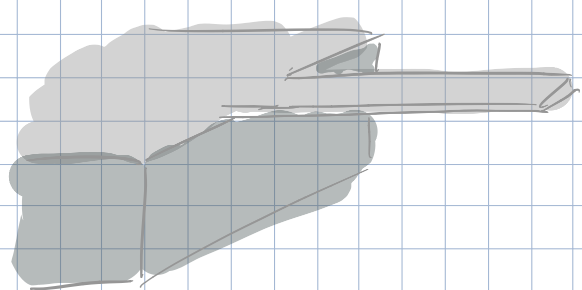
$$u_x(z) = A e^{i\beta z} + B e^{-i\beta z} + C e^{\beta z} + D e^{-\beta z}$$

or equivalently

$$u_x(z) = a\cos(\beta z) + b\sin(\beta z) + c\cosh(\beta z) + d\sinh(\beta z)$$

We can now apply boundary conditions:

Take, for instance, a singly-clamped beam (i.e. a cantilever)



(1) $u_x(0) = 0 \leftarrow$ no displacement at clamp

(2) $\frac{du_x}{dz}(0) = 0 \leftarrow$ no bending at clamp

(3) $\frac{d^2 u_x}{dz^2}(\ell) = 0 \leftarrow$ no torque at free end

(4) $\frac{d^3 u_x}{dz^3}(\ell) = 0 \leftarrow$ no net force over full beam

These conditions give : $a = -c$

$b = -d$

and

$$\cos(\beta l)\cosh(\beta l) + 1 = 0$$

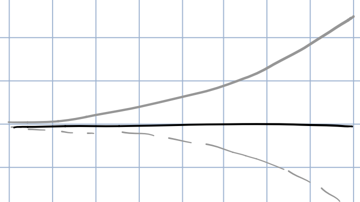
$$\beta_n l = 1.875,\ 4.694,\ 7.855,\ 10.996,\ \dots$$

$\therefore$

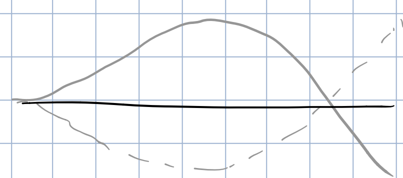
$$u_{xn}(z) = a_n\left[\cos(\beta_n z) - \cosh(\beta_n z)\right] + b_n\left[\sin(\beta_n z) - \sinh(\beta_n z)\right]$$

with $\dfrac{a_n}{b_n} = -1.362,\ -0.982,\ -1.008,\ -1.000,\ \dots$

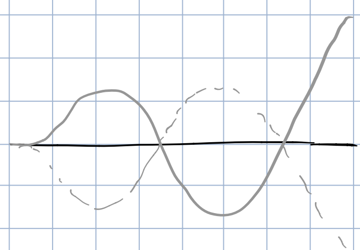
$n = 1$



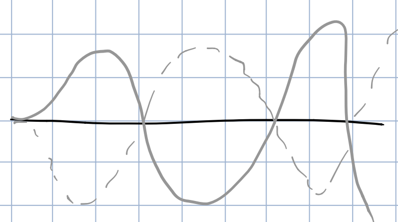
$n = 2$



$n = 3$



$n = 4$



$$\omega_n = \sqrt{\frac{EI_y}{\rho A}}\,\beta_n^2$$

II. Zener's Model of an Anelastic Solid

(F.N., p.282)

strain

$$\sigma = E\,\varepsilon$$

stress

Young's Modulus

$$\sigma + T_\varepsilon \frac{d\sigma}{dt} = E_R\left(\varepsilon + T_\sigma \frac{d\varepsilon}{dt}\right)$$

Consider harmonic motion:

$$\sigma = \sigma_0 e^{i\omega t}$$

$$\varepsilon = \varepsilon_0 e^{i\omega t}$$

$$\frac{\sigma_0}{\varepsilon_0} = E_R\left(\frac{1 + T_\sigma i\omega}{1 + T_\varepsilon i\omega}\right) \equiv E(\omega)$$

$$T = \sqrt{T_\sigma T_\varepsilon}$$

$$E(\omega) = \left(\frac{1+\omega^2 T^2}{1+\omega^2 T_\varepsilon^2} + i\,\frac{\omega T}{1+\omega^2 T_\varepsilon^2}\Delta\right) E_R$$

$$\Delta \equiv \frac{T_\sigma - T_\varepsilon}{T}$$

$$E_{eff}(\omega) = \frac{1+\omega^2 T^2}{1+\omega^2 T_\varepsilon^2} E_R$$

$$E(\omega) = E_{eff}(\omega)\left[1 + i\,\frac{\omega T}{1+\omega^2 T^2}\Delta\right]$$

Define :

$$\frac{1}{Q} \equiv \frac{\omega T}{1+\omega^2 T^2}\Delta$$

$$E(\omega) = E_{eff}(\omega)\left[1 + \frac{i}{Q}\right]$$

Back to Euler Bernoulli: :

$$E I_Y \frac{\partial^4 u_x}{\partial z^4} + \rho A \frac{\partial^2 u_x}{\partial t^2} = 0$$

$$\frac{d^4 u_x}{d z^4}(z) = \underbrace{\left(\frac{\rho A}{E_{eff}\left(1+\frac{i}{Q}\right) I_Y}\right)\omega^2}_{\beta^4} u_x(z)$$

Now :

$$\boxed{\begin{aligned} \omega_n' &= \sqrt{\frac{E_{eff} I_Y}{\rho A}}\,\beta_n^2\left(1 + \frac{i}{2Q}\right) \\ \omega_n' &= \left(1 + \frac{i}{2Q}\right)\omega_n \qquad \text{for } Q \gg 1 \end{aligned}}$$