Lecture 3 (30.09.2020)

Cantilever Dynamics

I. Cantilevers as Harmonic Oscillators

Today let's start by adding a driving force at to our beam dynamics analysis:

$$E I_y \frac{\partial^4 u_x}{\partial z^4} + \rho A \frac{\partial^2 u_x}{\partial t^2} = f(t)$$

$\left[\frac{\text{Force}}{\text{length}}\right]$

If we Fourier decompose $f(t)$, we can do the same for u_x, considering:

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega) e^{i\omega t} d\omega$$

$$u_x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_x(\omega) e^{i\omega t} d\omega$$

Plugging in these expressions and canceling out both the time dependence and integrals from the equations, we have:

$$E I_y \frac{\partial^4 \hat{U}_x}{\partial z^4} - \omega^2 \rho A \hat{U}_x = \hat{F}(\omega)$$

$\hat{U}_x$ can be written in terms of the eigenfunctions of the beam:

$$\hat{U}_x = \sum_{n=1}^{\infty} a_n u_{xn}$$

where $$\underbrace{\int_0^l u_{xn} u_{xm} \, dz = l^3 \delta_{mn}}_{\text{orthogonality}}$$

Then :

$$EI_y \sum_{n=1}^{\infty} a_n \underbrace{\frac{\partial^4 u_{xn}}{\partial z^4}} - \omega^2 \rho A \sum_{n=1}^{\infty} a_n u_{xn} = \hat{F}(\omega)$$

From before

$$\frac{\partial^4 u_{xn}}{\partial z^4} = \left(\frac{\rho A}{EI_y}\right) \omega_n^2 u_{xn}$$

$$\rho A \sum_{n=1}^{\infty} a_n \omega_n'^2 u_{xn} - \rho A \omega^2 \sum_{n=1}^{\infty} a_n u_{xn} = \hat{F}(\omega)$$

Integrating through with u_{xm} :

$$\rho A \left[\sum_{n=1}^{\infty} a_n \omega_n'^2 \cancelto{l^3 \delta_{nm}}{\int_0^l u_{xm} u_{xn} \, dz} - \omega^2 \sum_{n=1}^{\infty} a_n \cancelto{l^3 \delta_{nm}}{\int_0^l u_{xm} u_{xn} \, dz} \right]$$

$$= \int_0^l u_{xm} \hat{F}(\omega) \, dz$$

$$\rho A l^3 a_n \left(\omega_n'^2 - \omega^2 \right) = \int_0^l u_{xn} \hat{F}(\omega) \, dz$$

$$a_n = \frac{1}{\underbrace{\rho A \ell^3}_{m\ell^2}} \frac{1}{(\underbrace{\omega_n'^2}_{} - \omega^2)} \int_0^\ell u_{xn} \hat{F}(\omega)\, dz$$

$$\omega_n' = \left(1 + \frac{i}{2Q}\right) \omega_n$$

In the limit of high Q (small dissipation)

$$\boxed{a_n = \frac{1}{m\ell^2} \int_0^\ell u_{xn} \hat{F}(\omega)\, dz \left(\frac{1}{\omega_n^2 - \omega^2 + i \frac{\omega_n^2}{Q}} \right)}$$

The response of each mode n – as we will see – looks a lot like the response of a simple driven damped harmonic oscillator.

Let's now specify that our driving force $f(t)$ is a point force applied at $z = L$.

$$\therefore \hat{F}(\omega) = \hat{F}_p(\omega)\, \delta(z - \ell)$$

$$\int_0^\ell u_{xn} \hat{F}(\omega)\, dz = u_{xn}(\ell)\, \hat{F}_p(\omega)$$

If we now consider the response of the fundamental mode, $n = 1$, for our normalization we know that $u_{x1}(\ell) \simeq 2\ell$.

$\therefore$

$$a_1 = \frac{1}{m l^2} \cdot 2 l \hat{F}_p(\omega) \cdot \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}}$$

$$a_1 = \frac{2 \hat{F}_p(\omega)}{m l} \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}}$$

Driven displacement of $n=1$ mode at end:

$$a_1 u_{x,1}(l) = \frac{4 \hat{F}_p(\omega)}{m} \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}}$$

If ω is close to ω_1 and no other mode resonances, then we can say that the Fourier component of the cantilever's end deflection is:

$$\hat{x}_{end}(\omega) = a_1 u_{x,1}(l) = \frac{4 \hat{F}_p(\omega)}{m} \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}}$$

$$\boxed{\hat{x}_{end}(\omega) = \frac{\hat{F}_p(\omega)}{m_{eff}} \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}} \qquad \text{w/ } m_{eff} = \frac{m}{4}}$$

This expression can be rewritten in terms of a mechanical susceptibility $\chi_m(\omega)$:

$$\hat{x}_{end}(\omega) = \hat{F}_p(\omega)\,\chi_m(\omega)$$

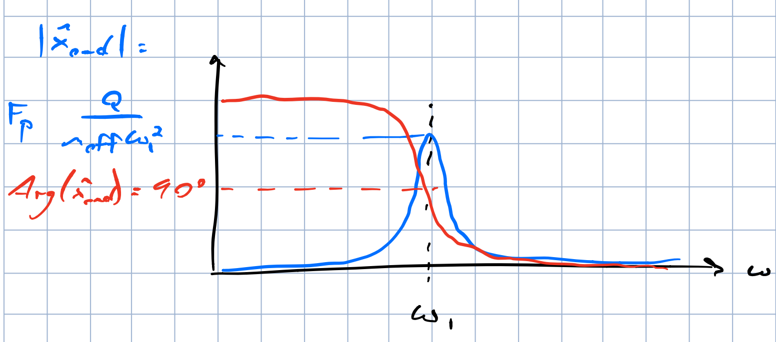
$$\text{w/} \quad \chi_m(\omega) = \frac{1}{m_{eff}} \frac{1}{\omega_1^2 - \omega^2 + i\frac{\omega_1^2}{Q}}$$

The cantilever behaves as a resonator, responding only to forces at frequency ω_1 within a bandwidth proportional to $1/Q$.

$$\chi_m(\omega_1) = -i\,\frac{Q}{m_{eff}\,\omega_1^2}$$

response multiplied by Q

$-90°$ phase shift



Recall what we derived for the static case: for a point force applied to the end, a cantilever bends:

$$u_x(z) = \frac{F_p}{2EI_y}\left(\ell z^2 - \frac{z^3}{3}\right)$$

$$u_x(\ell) = F_p \frac{\ell^3}{3EI_y} \qquad\qquad I_y = \frac{w t^3}{12}$$

$$u_x(\ell) = F_p\left(\frac{4\ell^3}{Ewt^3}\right)$$

$$\therefore \quad F_p = k_s \cdot u_x(\ell)$$

$$\boxed{\begin{array}{c}\underline{\text{Static spring const.}} \\ k_s = \dfrac{Ewt^3}{4\ell^3}\end{array}}$$

In dynamic case :

$$\hat{F}_p(\omega) = \frac{1}{\mathcal{X}_m(\omega)}\,\hat{x}_{end}(\omega)$$

$$k_D = \frac{1}{\mathcal{X}(\omega)}$$

$$\omega_1 = \sqrt{\frac{EI_z}{\rho A}}\,\beta_n^2$$

On resonance :

$$k_D = \frac{1}{\mathcal{X}(\omega_1)} = i\,\frac{m_{eff}\,\omega_1^2}{Q} = i\,\frac{\cancel{\rho wt}\ell}{4Q}\,\frac{EI_z}{\cancel{\rho wt}}\,\frac{1.875^4}{\ell^{\cancel{4}\,3}}$$

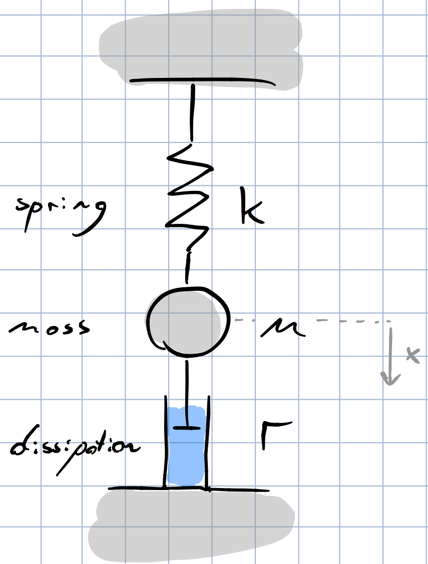
$$k_D = \frac{i}{Q}\cdot\frac{Ewt^3}{48\,\ell^3}\cdot\underbrace{1.875^4}_{12.36} \approx \frac{i}{Q}\,\frac{Ewt^3}{4\,\ell^3} = \frac{i}{Q}\,k_s$$

$$\therefore \quad \boxed{\begin{array}{c}\underline{\text{Dynamic Spring const.}} \\ k_D \approx \dfrac{i}{Q}\,k_s\end{array}}$$

Softer by factor Q

II. Simple Harmonic Oscillator

Let's consider for a moment the simple harmonic oscillator:



$$m\ddot{x} + \Gamma\dot{x} + kx = F(t)$$

Let's define the Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\omega)\, e^{i\omega t}\, d\omega$$

If we take the Fourier transform of the equation of motion, we have

$$-m\omega^2\hat{x}(\omega) + i\Gamma\omega\hat{x}(\omega) + kx(\omega) = \hat{F}(\omega)$$

$$\hat{x}(\omega) = \frac{\hat{F}(\omega)}{m}\frac{1}{\frac{k}{m} - \omega^2 + i\Gamma\omega}$$

Define: $k = m\omega_0^2$

$\Gamma = \frac{m\omega_0}{Q}$

$$\boxed{\hat{x}(\omega) = \frac{\hat{F}(\omega)}{m}\frac{1}{\omega_0^2 - \omega^2 + i\frac{\omega_0\omega}{Q}}}$$

This expression is very similar to that of a driven damped beam. Here, again, we can define a susceptibility:

$$\hat{x}(\omega) = \hat{F}(\omega)\, \chi_m(\omega)$$

w/ $$\chi_m(\omega) = \frac{1}{m} \frac{1}{\omega_0^2 - \omega^2 + i\frac{\omega_0 \omega}{Q}}$$

On resonance, the harmonic oscillator reacts,

$$\chi_m(\omega_0) = -i \frac{Q}{m\omega_0^2}$$

response multiplied by Q

-90° phase shift

Similarly the dynamic spring constant is softened by Q:

$$k_D = \frac{1}{\chi_m(\omega_0)} = i \frac{m\omega_0^2}{Q}$$

$$k_D = i \frac{k}{Q}$$

So, the beam and the harmonic oscillator behave almost the same (except for difference in Q term, which is important only off resonance:

$i\frac{\omega_0 \omega}{Q}$ vs. $i\frac{\omega_1^2}{Q}$.