Lecture 4 (07.10.2020)

Dissipation and Noise

I. Power Spectral Density

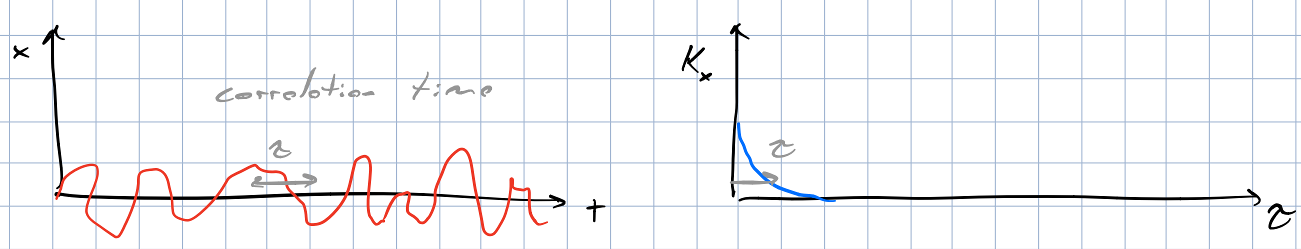
We now know how beams — and by extension other continuous objects like strings, membranes, and rods — behave under both static and dynamic forces. We find that they act as transducers and as resonators with geometry and losses determining their properties.

We have written transfer functions and can solve for their response to time-varying forces. Now we must consider the effects of fluctuations ...

First, however, we have to discuss how to quantify fluctuations. For this, we begin by defining the correlation function:

$$K_x(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{x(t)\, x^*(t-\tau)}{T}\, dt$$

The above function measures how well correlated fluctuations are after a time τ.



If we write the integrand in terms of the Fourier transforms of $x(t)$ and $x^*(t)$:

$$K_x(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\hat{x}(\omega)\,\hat{x}^*(\omega')}{T} e^{i(\omega-\omega')t} e^{i\omega'\tau}\, d\omega\, d\omega'\, dt$$

Recall :

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}\, dt$$

$$K_x(\tau) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\hat{x}(\omega)\,\hat{x}^*(\omega')}{T} \delta(\omega-\omega')\, e^{i\omega'\tau}\, d\omega\, d\omega'$$

$$K_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} \frac{\hat{x}(\omega)\,\hat{x}^*(\omega)}{T}} e^{i\omega\tau}\, d\omega$$

$$\boxed{S_x(\omega) = \lim_{T\to\infty} \frac{\hat{x}(\omega)\,\hat{x}^*(\omega)}{T}}$$

$$\boxed{K_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega}$$

Correlation function and Spectral Density are F.T. pairs.

This also means that :

$$K_x(0) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{x(t)\,x^*(t)}{T}\, dt = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{|x(t)|^2}{T}\, dt = \langle x^2 \rangle$$

$$K_x(0) = \langle x^2 \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega$$

Power spectral density (PSD) $\left[\frac{m^2}{Hz}\right]$

The root-mean-square fluctuations can be expressed as:

$$x_{rms} = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega}$$

What is $S_x(\omega)$ for a beam? Well, we take the definition of the PSD and express it in terms of the F.T.s.

Beam:

$$S_{x_{end}}(\omega) = \lim_{T\to\infty} \frac{\hat{x}_{end}(\omega)\, \hat{x}^*_{end}(\omega)}{T}$$

$$S_{x_{end}}(\omega) = \lim_{T\to\infty} \frac{F_p(\omega)\, F_p^*(\omega)}{T} \frac{1}{m_{eff}^2} \frac{1}{(\omega_1^2 - \omega^2)^2 + \frac{\omega_1^4}{Q^2}}$$

$$S_{x_{end}}(\omega) = \frac{S_{F_p}(\omega)}{m_{eff}^2} \left(\frac{1}{(\omega_1^2 - \omega^2)^2 + \frac{\omega_1^4}{Q^2}} \right)$$

Similarly:

Harmonic Osc.:

Force Flucts. (S_F) transduced into displacement Flucts. (S_x)

$$S_x(\omega) = \lim_{T\to\infty} \frac{\hat{x}(\omega)\,\hat{x}^*(\omega)}{T} =$$

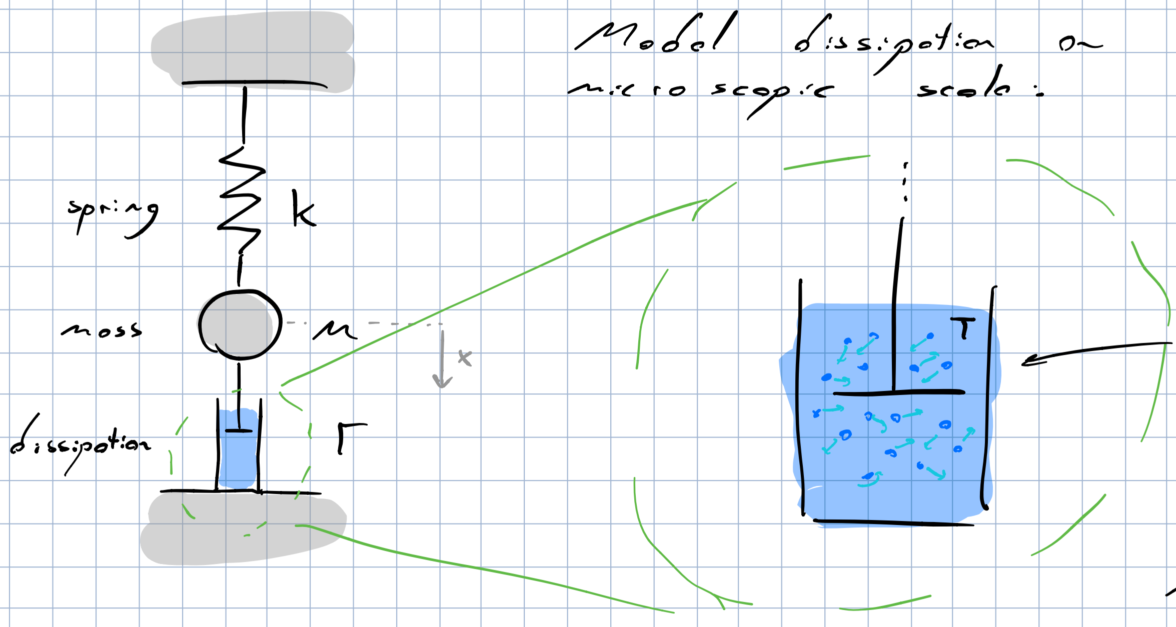
$$\boxed{S_x(\omega) = \frac{S_F(\omega)}{m^2}\left(\frac{1}{(\omega_0^2-\omega^2)^2 + \frac{\omega_0^2\omega^2}{Q^2}}\right)}$$

Near resonance, i.e. $\omega_0 = \omega_1 \sim \omega$, the beam and the harmonic oscillator respond the same to fluctuations and drives. From now on we will use this as an excuse to approximate the modes of beams, membranes etc. as simple harmonic oscillators.

II. Fluctuation - Dissipation

In order to understand the motion of our resonator, we have to understand the force fluctuations that drive it: $S_F(\omega)$. These, it turns out, are tightly related to the dissipation Γ, which the resonator experiences. Let's consider the harmonic oscillator ...

Model dissipation on microscopic scale:



Random force $f(t)$ acts on mass due to particles

$$m\ddot{x} + kx = F(t) + f(t)$$

force from microscopic collisions

external drive force

$$m\ddot{x} = \underbrace{F(t) - kx}_{\mathcal{F}(t)} + \underbrace{f(t)}_{\text{random force}}$$

$\mathcal{F}(t)$: slowly varying external forces

random force due to microscopic collisions with heat reservoir at T

We cannot know $f(t)$ fully, only statistically.
$f(t)$ has a correlation time τ^* which is very short ($\sim 10^{-13}$ s for a typical liquid).
Consider a macroscopically short time τ such that $\tau \gg \tau^*$:

$$m\left(v(t+\tau) - v(t)\right) = \mathcal{F}(t)\,\tau + \int_t^{t+\tau} f(t')\,dt'$$

Taking an average over the ensemble:

$$m\langle v(t+\tau) - v(t)\rangle = \widetilde{\mathcal{F}}(t)\,\tau + \int_t^{t+\tau} \langle F(t')\rangle\, dt' \qquad (1)$$

Let's consider our small system A within a larger heat bath B at temperature T. The probability of A being in some state r, W_r, is proportional to the corresponding number of states available to B, Ω.

At time t:

energy of bath

$$W_r(t) \propto \Omega(E')$$

energy increase of bath

$$W_r(t+\tau') \propto \Omega(E' + \Delta E')$$

$\tau' > \tau^*$, i.e. any accessible state is equally likely

From statistical mechanics:

$$\frac{W_r(t+\tau')}{W_r(t)} = \frac{\Omega(E'+\Delta E')}{\Omega(E')} = e^{\frac{\Delta E'}{k_B T}}$$

In other words the probability that A is found in a given state r at some later time is increased, if more energy becomes available to B (heat reservoir).

For a small change

$$W_r(t+\tau') = W_r(t)\, e^{\frac{\Delta E'}{k_B T}} \approx W_r(t)\left(1 + \frac{\Delta E'}{k_B T}\right)$$

$$\langle f(t+\tau')\rangle = \sum_r W_r(t+\tau') f_r = \sum_r W_r(t)\left(1 + \frac{\Delta E'}{k_B T}\right) f_r$$

$$\langle f(\underbrace{t+\tau'}_{t'})\rangle = \langle f(t)\rangle + \frac{1}{k_B T}\langle \Delta E' f(t)\rangle$$

$$\langle f(t')\rangle = \underbrace{\langle f(t)\rangle}_{0 \;\leftarrow\; \text{zero mean force (random)}} + \frac{1}{k_B T}\langle \Delta E' f(t)\rangle$$

$$\Delta E' = -\int_t^{t'} v(t'') f(t'')\, dt'' \approx -v(t)\int_t^{t'} f(t'')\, dt''$$

$$\langle f(t')\rangle = \frac{1}{k_B T}\left\langle -v(t)\int_t^{t'} f(t'')\, dt'' \; f(t)\right\rangle$$

$$= -\frac{\langle v(t)\rangle}{k_B T}\int_t^{t'} \langle f(t'') f(t)\rangle\, dt''$$

$$\langle f(t')\rangle = \frac{\langle v(t)\rangle}{k_B T}\int_0^{t-t'} \langle f(t) f(t-s)\rangle\, ds$$

$s = t - t''$; $t'' = t - s$

$ds = -dt''$

$$\int_t^{t+\tau} \langle f(t')\rangle\, dt' = -\frac{\langle v(t)\rangle}{k_B T}\int_t^{t+\tau} dt' \int_{t-t'}^{0} ds\, \underbrace{\langle f(t) f(t-s)\rangle}_{K_f(s)\,:\ \text{correlation}} \qquad (2)$$

Function

One aside about the correlation function: $K_f(s)$ is independent of t.

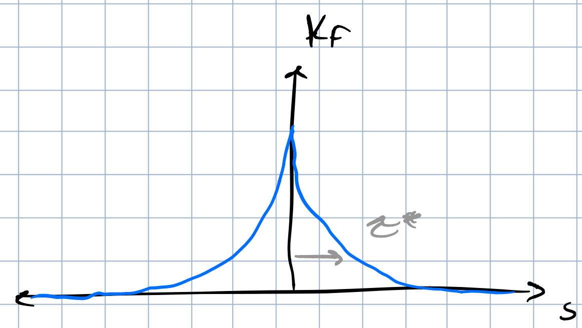
$$K_f(s) = \langle f(t)\, f(t-s) \rangle = \langle f(t_1)\, f(t_1 - s) \rangle$$

if $t_1 = t+s$:

$$K_f(s) = \langle f(t)\, f(t-s) \rangle = \langle f(t+s)\, f(t) \rangle$$

The correlation function is symmetric $\rightarrow$ $\underline{K_f(s) = K_f(-s)}$

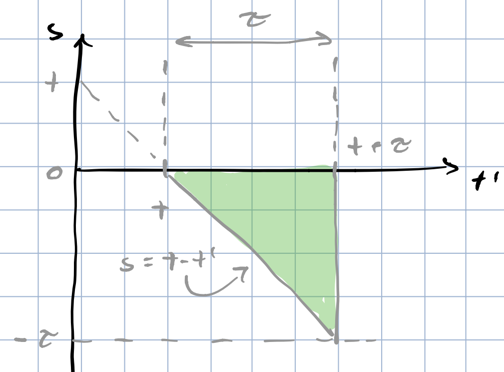
It drops to zero for $s \sim \tau^*$ $\rightarrow$



Back to (1) and (2):

$$m\langle v(t+\tau) - v(t) \rangle = \mathcal{F}(t)\,\tau - \frac{\langle v(t) \rangle}{k_B T} \underbrace{\int_t^{t+\tau} dt' \int_{t-t'}^{0} K_f(s)\, ds}_{}$$

Domain of integration



$$I = \int_t^{t+\tau} dt' \int_{t-t'}^{0} ds\, K_f(s)$$

$$I = \int_{-\tau}^{0} ds \int_{t-s}^{t+\tau} dt'\, K_f(s)$$

$$I = \int_{-\tau}^{0} ds\ (\tau + s)\ K_f(s)$$

Recall : $\tau >> \tau^*$ and $K(s) \rightarrow 0$ for $|s| >> \tau^*$

$$\therefore \quad I \approx \tau \int_{-\infty}^{0} ds\ K_f(s) = \frac{\tau}{2} \int_{-\infty}^{\infty} ds\ K_f(s)$$

$K_f(s)$ is symmetric

So :

$$\boxed{m \langle v(t+\tau) - v(t) \rangle = \mathcal{F}(t)\,\tau - \frac{\langle v(t) \rangle}{2 k_B T}\,\tau \int_{-\infty}^{\infty} K_f(s)\, ds}$$

Since $\langle v(t) \rangle$ varies slowly over τ :

$$m \frac{d\langle v(t) \rangle}{dt} = m \frac{\langle v(t+\tau) \rangle - \langle v(t) \rangle}{\tau}$$

$$\therefore \quad m \frac{d\langle v(t) \rangle}{dt} = \underbrace{\mathcal{F}(t)} - \frac{\langle v(t) \rangle}{2 k_B T} \int_{-\infty}^{\infty} K_f(s)\, ds$$

Recall : $\mathcal{F}(t) = F(t) - kx$

<u>Macroscopically</u> :

$$m \ddot{x} = F(t) - kx - \frac{\dot{x}}{2 k_B T} \int_{-\infty}^{\infty} K_f(s)\, ds$$

$$(3) \quad \boxed{m\ddot{x} + \underbrace{\left[\frac{1}{2k_BT}\int_{-\infty}^{\infty} K_f(s)\,ds\right]} \dot{x} + kx = F(t)}$$

This is the dissipation Γ !

Connection between fluctuating forces $f(t)$ and dissipation Γ is :

$$\Gamma = \frac{1}{2k_BT}\int_{-\infty}^{\infty} K_f(s)\,ds$$

Since the fluctuating forces are nearly uncorrelated on macroscopic time-scales ($\tau^* \ll \tau$), we can approximate :

$$K_f(s) \approx f_0^2\,\delta(s)$$

$$\therefore \quad \Gamma = \frac{1}{2k_BT} f_0^2 \quad \longrightarrow \quad f_0^2 = 2k_BT\Gamma$$

$$\longrightarrow \quad \boxed{K_f(s) = 2k_BT\Gamma\,\delta(s)}$$

In this way we retrieve the original equation of motion from (3) :

$$m\ddot{x} + \Gamma\dot{x} + kx = F(t)$$

This also allows us to find the PSD of the fluctuating forces $f(t)$:

$$S_f(\omega) = \int_{-\infty}^{\infty} K_f(s)\, e^{-i\omega s}\, ds$$

$$\boxed{S_f(\omega) = 2k_BT\Gamma}$$

$\leftarrow$ Double-sided $-\infty < \omega < \infty$

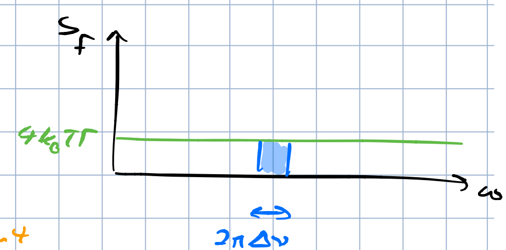
This spectral density is constant in ω (white)

For real-valued signals $S(\omega)$ is even $\left(S(\omega) = S(-\omega)\right)$. Therefore it is sometimes useful to define a single-sided PSD for only positive ω :

$$\bar{S}(\omega) = S(\omega) + S(-\omega)$$

$$\therefore \quad \boxed{\bar{S}_f(\omega) = 4k_BT\Gamma}$$

$\leftarrow$ Single-sided $0 < \omega < \infty$



$$F_{min} = \sqrt{4k_BT\Gamma\Delta\nu}$$

thermal force and therefore minimum detectable force

measurement bandwidth

Size of thermal displacement fluctuations:

$$\langle x^2 \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega$$

$$\langle x^2 \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_F(\omega)}{m^2}\left(\frac{1}{(\omega_0^2-\omega^2)^2 + \frac{\omega_0^2\omega^2}{Q^2}}\right) d\omega$$

$$\langle x^2 \rangle = \frac{k_B T \Gamma}{\pi m^2}\underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{(\omega_0^2-\omega^2)^2 + \frac{\Gamma^2\omega^2}{m^2}}}_{\frac{\pi m}{\omega_0^2 \Gamma}}$$

$$\boxed{\Gamma = \frac{m\omega_0}{Q} \qquad k = m\omega_0^2}$$

$$\langle x^2 \rangle = \frac{k_B T \cancel{\Gamma}}{\cancel{\pi} m^{\cancel{2}}}\,\frac{\cancel{\pi}\,\cancel{m}}{\omega_0^2\cancel{\Gamma}} = \frac{k_B T}{m\omega_0^2} = \frac{k_B T}{k}$$

$$\boxed{\frac{1}{2}k\langle x^2 \rangle = \frac{1}{2}k_B T}$$

← We retrieve the equipartition theorem!