Lecture 5 (14.10.2020)

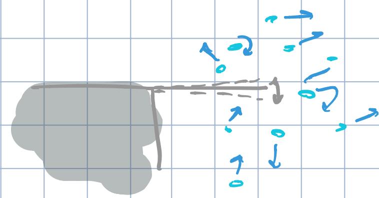
Dissipation and Noise

I. Review and Analogy to Electronics

A quick review:

Langevin Equation for a nanomechanical resonator

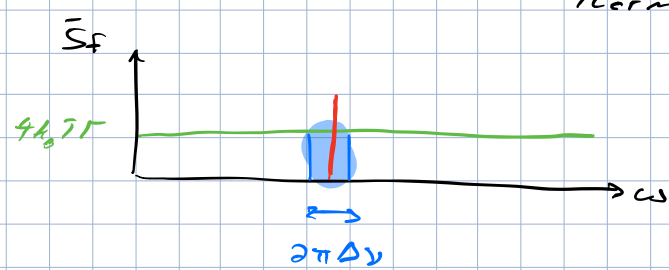
$$m\ddot{x} + \Gamma\dot{x} + kx = F(t)$$



single-sided

$$\boxed{\bar{S}_F(\omega) = 4k_BT\Gamma}$$

Spectral density of thermal forces



Thermal force sets minimum measurable force:

$$\boxed{F_{min} = \sqrt{4k_BT\Gamma\Delta\nu}}$$

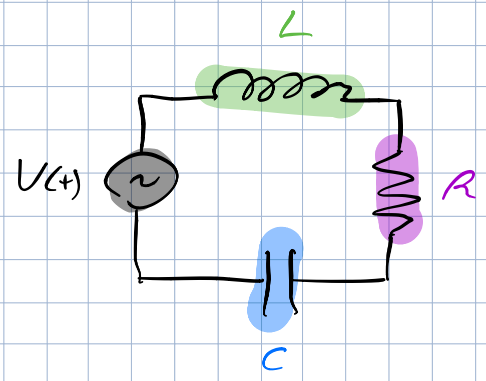
These random force fluctuations drive random displacement fluctuations w/ a spectrum:

$$\boxed{S_x(\omega) = \frac{2k_BT\Gamma}{m^2}\frac{1}{(\omega_0^2-\omega^2)^2 + \frac{\Gamma^2\omega^2}{m^2}}}$$

The Langevin equation, which we have been discussing is not the only one. Take for example the standard description of a linear electronic circuit:

$$\underbrace{-L\ddot{Q}}_{V_L = -L\frac{dI}{dt}} + \underbrace{R\dot{Q}}_{V_R = RI} + \underbrace{\frac{1}{C}Q}_{V_C = \frac{Q}{C}} = V(t)$$

$$\frac{dI}{dt} = \ddot{Q} \qquad I = \dot{Q}$$



The PSD of thermal voltage fluctuations is proportional to T and R:

$$\bar{S}_V(\omega) = 4 k_B T R$$

Johnson Noise

$$V_{min} = \sqrt{4 k_B T R \Delta\nu}$$

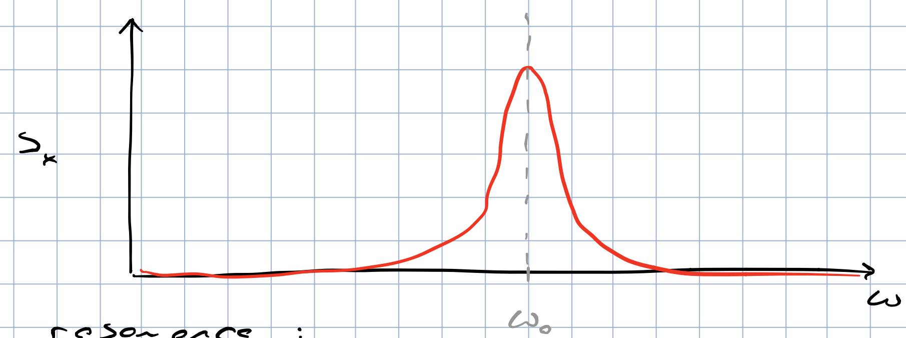
Johnson voltage noise in bandwidth $\Delta\nu$

II. Dissipation-induced Amplitude Noise

PSD of the displacement noise caused by thermal forces:

$$\boxed{S_x(\omega) = \frac{2 k_B T \Gamma}{m^2} \frac{1}{(\omega_0^2 - \omega^2)^2 + \frac{\Gamma^2 \omega^2}{m^2}}}$$

So even with a white force PSD (i.e. $S_f(\omega) = 2 k_B T \Gamma$), the displacement PSD has "color" in the sense that it has a frequency dependence imposed by the mechanical resonator.



On resonance:

$$S_x(\omega_0) = \frac{2 k_B T \cancel{\Gamma}}{\cancel{m^2}} \cdot \frac{\cancel{m^2}}{\Gamma^{\cancel{2}} \omega_0^2}$$

$$S_x(\omega_0) = \frac{2 k_B T}{\Gamma \omega_0^2}$$

For practical matters, let's consider the single-sided PSD:

$$\bar{S}_x(\omega) = \frac{4 k_B T \Gamma}{m^2} \frac{1}{(\omega_0^2 - \omega^2)^2 + \frac{\Gamma^2 \omega^2}{m^2}}$$

$$\bar{S}_x(\omega_0) = \frac{4 k_B T}{\Gamma \omega_0^2} = \frac{4 k_B T Q}{m \omega_0^3} \quad \left[\frac{m^2}{Hz} \right]$$

$$x_{min} = \frac{2}{\omega_0} \sqrt{\frac{k_B T}{\Gamma} \Delta\nu} \quad \left[\frac{m}{\sqrt{Hz}} \right]$$

Single-sided $0 < \omega < \infty$

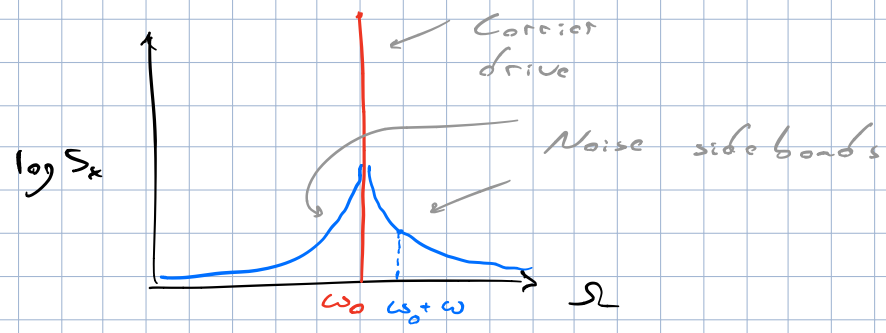
thermal displacement and therefore minimum detectable displacement

measurement bandwidth

III. Dissipation-induced Phase Noise

Thermal fluctuations will not only induce displacement noise (amplitude noise), but also phase noise. This is particularly important for time-keeping or for frequency-based measurements, as we will see.

Take a resonator that is driven by a carrier signal near its resonance frequency ω_0 and dissipation-induced thermal force noise: $S_F(\omega) = 2 k_B T \Gamma$.



Now we could represent the carrier at ω_0 and a single spectral component of the noise at ω as:

$$x(t) = \underline{x_0 \sin(\omega_0 t)} + \underline{x_n \sin\left((\omega_0 + \omega)t + \phi\right)}$$

$$x(t) = x_0 \sin(\omega_0 t) + x_n \sin(\omega_0 t)\cos(\omega t + \phi) + x_n \cos(\omega_0 t)\sin(\omega t + \phi)$$

$$x(t) = \underbrace{\left[x_0 + x_n \cos(\omega t + \phi)\right]}_{A} \sin(\omega_0 t) + \underbrace{\left[x_n \sin(\omega t + \phi)\right]}_{B} \cos(\omega_0 t)$$

$$x(t) = A \sin(\omega_0 t) + B\cos(\omega_0 t)$$

The resultant time-varying amplitude at ω_0 is:

$$R = \sqrt{A^2 + B^2} = \sqrt{x_0^2 + 2x_0 x_n \cos(\omega t + \phi) + x_n^2}$$

$$R = x_0 \sqrt{1 + \frac{2x_n}{x_0}\cos(\omega t + \phi) + \frac{x_n^2}{x_0^2}}$$

For a carrier drive much larger than the noise $x_0 \gg x_n$:

$$R \approx x_0 \left(1 + \frac{x_n}{x_0}\cos(\omega t + \phi)\right)$$

$$Q = x_0 \left(1 + \underbrace{\frac{x_n}{x_0} \sin\left(\omega t + \phi + \frac{\pi}{2}\right)}_{\text{Amplitude modulation}} \right)$$

$$\langle M^2 \rangle = \frac{x_n^2}{2 x_0^2}$$

The phase angle of $x(t)$ with respect to the pure carrier $\sin(\omega_0 t)$ is :

$$\tan\Theta = \frac{B}{A} = \frac{x_n \sin(\omega t + \phi)}{x_0 + x_n \cos(\omega t + \phi)}$$

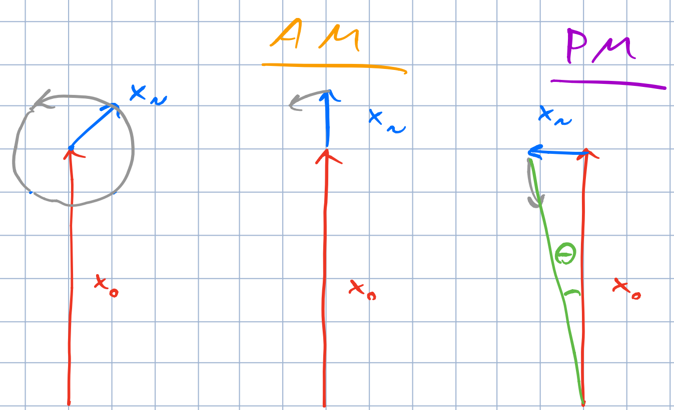
For $x_0 >> x_n$:

$$\tan\Theta \approx \frac{x_n}{x_0} \sin(\omega t + \phi)$$

$$\Theta \approx \underbrace{\frac{x_n}{x_0} \sin(\omega t + \phi)}_{\text{Phase modulation}} \qquad \langle \Theta^2 \rangle = \frac{x_n^2}{2 x_0^2}$$

Phasor Representation:

Noise rotating about carrier at ω

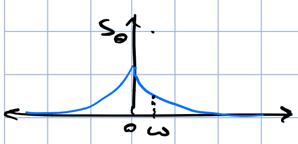


This means that noise in the sidebands of the carrier (at $\omega_0 + \omega$) can be represented as one half amplitude modulation noise and one half phase modulation noise. For frequency measurements or clocks, amplitude modulation is not important, only phase noise.

Averaging over time we can relate the phase fluctuations at ω with the displacement noise at $\omega_0 + \omega$:

$$\langle \theta^2 \rangle_\omega = \frac{1}{2} \frac{x_\omega^2}{x_0^2} = \frac{1}{x_0^2} \langle x^2 \rangle_{\omega_0+\omega}$$

Expressed in terms of PSDs :



$$\frac{1}{2\pi} S_\theta(\omega)\, d\omega = \frac{1}{x_0^2} \left[\frac{1}{2\pi} \bar{S}_x(\omega_0+\omega)\, d\omega \right]$$

$$\therefore \quad \boxed{S_\theta(\omega) = \frac{1}{x_0^2} \bar{S}_x(\omega_0 + \omega)} \quad \text{(double - sided)}$$

$$\boxed{\bar{S}_\theta(\omega) = \frac{2}{x_0^2} \bar{S}_x(\omega_0 + \omega)} \quad \text{(single - sided)}$$

If we now plug in our thermal PSD for $S_x(\omega)$, we can solve for the thermal phase PSD :

$$\Gamma = \frac{m\omega_0}{Q}$$

$$S_\theta(\omega) = \frac{4 k_B T \Gamma}{x_0^2 m^2} \frac{1}{(2\omega_0\omega + \omega^2)^2 + \frac{\Gamma^2(\omega_0+\omega)^2}{m^2}}$$

For frequencies that are well off resonance, $\omega \gg \frac{\omega_0}{Q}$, but small compared to the resonance frequency, $\omega \ll \omega_0$:

$$S_\theta(\omega) \approx \frac{k_B T \Gamma}{x_0^2 m^2} \frac{1}{\omega_0^2 \omega^2}$$

Double-sided : $S_\theta(\omega) \approx \dfrac{k_B T \Gamma}{x_0^2 m^2 \omega_0^2} \dfrac{1}{\omega^2} = \dfrac{k_B T}{x_0^2 m \omega_0 Q} \dfrac{1}{\omega^2}$

Single-sided : $\bar{S}_\theta(\omega) = \dfrac{2 k_B T \Gamma}{x_0^2 m^2 \omega_0^2} \dfrac{1}{\omega^2} = \dfrac{2 k_B T}{x_0^2 m \omega_0 Q} \dfrac{1}{\omega^2}$

IV. Dissipation-induced Frequency Noise

Frequency noise and phase noise are closely related. For a phase modulation θ at frequency ω, we have a frequency modulation $\delta\omega_0 = \omega\,\theta$.

$$\hookrightarrow \quad \delta\nu_0 = \frac{\omega}{2\pi}\,\theta$$

$$\therefore \quad S_\nu(\omega) = \left(\frac{\partial \delta\nu_0}{\partial\theta}\right)^2 S_\theta(\omega)$$

$$S_\nu(\omega) = \frac{\omega^2}{4\pi^2} S_\theta(\omega)$$

$$S_\nu(\omega) = \frac{k_B T \Gamma}{\pi^2 x_0^2 m^2} \frac{\omega^2}{(2\omega_0\omega + \omega^2)^2 + \frac{\Gamma^2(\omega_0+\omega)^2}{m^2}}$$

Again, in the limit $\omega \gg \frac{\omega_0}{Q}$ and $\omega \ll \omega_0$:

$$S_\nu(\omega) \approx \frac{k_B T \Gamma}{4\pi^2 x_0^2 m^2 \omega_0^2} = \frac{k_B T}{4\pi^2 x_0^2 m \omega_0 Q}$$

Single-sided : $\bar{S}_\nu(\omega) = \dfrac{k_B T \Gamma}{2\pi^2 x_0^2 m^2 \omega_0^2} = \dfrac{k_B T}{2\pi^2 x_0^2 m \omega_0 Q}$

white noise in this range

V. Measuring using a Harmonic Oscillator

Take a harmonic mechanical mode obeying the usual equation of motion:

$$m\ddot{x} + \Gamma\dot{x} + kx = 0 \,,$$

where no forces are applied. The solution to this differential equation has the form:

$$x(t) = A\, e^{i\omega t}$$

Therefore,

$$-m\omega^2 + i\Gamma\omega + k = 0$$

$$\omega = \frac{i\frac{\Gamma}{m} \pm \sqrt{\frac{4k}{m} - \frac{\Gamma^2}{m^2}}}{2}$$

$$\omega = i\frac{\Gamma}{2m} \pm \sqrt{\frac{k}{m} + \frac{\Gamma^2}{4m^2}}$$

These solutions correspond to decaying harmonic oscillations. If we define $\omega_0 = \sqrt{\frac{k}{m}}$ and $\Gamma = \frac{m\omega_0}{Q}$,

$$\omega = i\frac{\omega_0}{2Q} \pm \omega_0 \underbrace{\sqrt{1 + \frac{1}{4Q^2}}}$$

This term ≈ 1

For $Q >> 1$ i.e. small Γ.

Therefore for high Q, the resonant frequency is given by $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$.

$$x(t) = e^{-\frac{\omega_0}{2Q}t}\left(A e^{i\omega_0 t} + B e^{-i\omega_0 t}\right)$$

The resonator's resonance frequency can be used to measure changes in mass, spring constant, and force gradient.

— Measuring Mass

Let's take a small change in mass δm:

$$\omega_0 + \delta\omega_0 = \sqrt{\frac{k}{m+\delta m}}$$

$$\omega_0 + \delta\omega_0 = \omega_0\sqrt{\frac{m}{m+\delta m}}$$

$$\omega_0 + \delta\omega_0 = \omega_0\left(1 + \frac{\delta m}{m}\right)^{-1/2}$$

Expanding around small $\frac{\delta m}{m}$ and keeping only first order terms:

$$\omega_0 + \delta\omega_0 \approx \omega_0\left(1 - \frac{1}{2}\frac{\delta m}{m}\right)$$

$$\frac{\delta\omega_0}{\omega_0} = -\frac{1}{2}\frac{\delta m}{m}$$

$$\boxed{\frac{\delta\nu_0}{\nu_0} = -\frac{1}{2}\frac{\delta m}{m}}$$

← Relative frequency shift is proportional to relative mass change.

Since we now know the relationship between changes in resonance frequency and changes in mass, we can now write down a thermal limit a minimum detectable mass based on thermal frequency fluctuations $S_\nu(\omega)$. From above:

$$\delta m = \frac{-2m}{\nu_0}\delta\nu_0 = \frac{-4\pi m}{\omega_0}\delta\nu_0$$

$$\bar{S}_m(\omega) = \left(\frac{-4\pi m}{\omega_0}\right)^2 \bar{S}_\nu(\omega)$$

$$\bar{S}_m(\omega) = \frac{16\pi^2 m^2}{\omega_0^2}\bar{S}_\nu(\omega)$$

Again, in the limit $\omega \gg \frac{\omega_0}{Q}$ and $\omega \ll \omega_0$:

$$\bar{S}_m(\omega) = \frac{\overset{8}{\cancel{16}}\cancel{\pi^2}\cancel{m^2}}{\omega_0^2}\frac{k_B T \Gamma}{\cancel{2\pi^2} x_0^2 \cancel{m^2}\omega_0^2} = \frac{8k_B T\Gamma}{x_0^2\omega_0^4} = \frac{8k_B T m}{x_0^2\omega_0^3 Q}$$

$$\boxed{\bar{S}_m(\omega) = \frac{2}{x_0^2\omega_0^4}\bar{S}_F}$$

← PSD of thermal "mass" fluctuations

As a result the minimum detectable mass due to thermal fluctuations is:

$$m_{min} = \sqrt{\frac{8 k_B T \Gamma \Delta\nu}{x_0^2 \omega_0^4}} = \frac{2}{x_0 \omega_0^2} \sqrt{2 k_B T \Gamma \Delta\nu} \quad [kg]$$

measurement bandwidth

To improve sensitivity (i.e. make m_{min} smaller), we have to reduce T, Γ, $\Delta\nu$ and increase x_0 and ω_0.

$\longrightarrow$ Cold, high-frequency, low-loss, resonators are best.

— Measuring Force Gradients

Similarly, the system can be exposed to an external force gradient, $\frac{\partial F}{\partial x}$, or equivalently a change in spring constant δk. In each case the equation of motion becomes:

$$m\ddot{x} + \Gamma \dot{x} + kx = \frac{\partial F}{\partial x} x$$

or

$$m\ddot{x} + \Gamma \dot{x} + (k + \delta k) x = 0$$

In both cases the analysis is the same, so we will proceed with δk ($\delta k = -\frac{\partial F}{\partial x}$).

$$\omega_0 + \delta\omega_0 = \sqrt{\frac{k + \delta k}{m}}$$

$$\omega_0 + \delta\omega_0 = \omega_0 \left(1 + \frac{\delta k}{k}\right)^{1/2}$$

For small $\frac{\delta k}{k}$:

$$\omega_0 + \delta\omega_0 \approx \omega_0 \left(1 + \frac{1}{2}\frac{\delta k}{k}\right)$$

$$\boxed{\begin{aligned} \frac{\delta\omega_0}{\omega_0} &\approx \frac{1}{2}\frac{\delta k}{k} \\ \frac{\delta\omega_0}{\omega_0} &= -\frac{1}{2}\frac{\left(\frac{\partial F}{\partial x}\right)}{k} \end{aligned}}$$

← Relative frequency shift is also proportional to force gradients and changes in stiffness.

Now that we have the relationship between changes in k (or external gradients) and frequency, we can state the thermomechanical limits imposed by frequency noise $S_\nu(\omega)$.

$$\delta k = \frac{2k}{\nu_0}\delta\nu_0 = \frac{4\pi k}{\omega_0}\delta\nu_0 = 4\pi m\omega_0 \delta\nu_0$$

$$\bar{S}_k(\omega) = 16\pi^2 m^2 \omega_0^2 \, \bar{S}_\nu(\omega)$$

In the limit $\omega \gg \frac{\omega_0}{Q}$ and $\omega \ll \omega_0$:

$$\bar{S}_k(\omega) = \cancel{16}^{\,8}\cancel{\pi^2}\cancel{m^2}\cancel{\omega_0^2} \frac{k_B T \Gamma}{\cancel{2\pi^2} x_0^2 \cancel{m^2} \cancel{\omega_0^2}} = \frac{8 k_B T \Gamma}{x_0^2}$$

$$\boxed{\bar{S}_k(\omega) = \frac{2}{x_0^2} \bar{S}_f}$$

← PSD of thermal "spring constant" or force gradient fluctuations

As a result, the minimum detectable spring constant change or force gradient due to thermal fluctuations is :

$$k_{min} = \frac{2}{x_0} \sqrt{2 k_B T \Gamma \Delta\nu} \qquad \left[\frac{N}{m}\right]$$

measurement bandwidth

To improve sensitivity, the same measures should be taken as for optimizing F_{min} : decrease T, Γ, $\Delta\nu$. In addition the driving amplitude x_0 should be increased as much as possible without affecting spatial resolution and without entering a non-linear regime.