Lecture 8 (04.11.2020)

Feedback Cooling

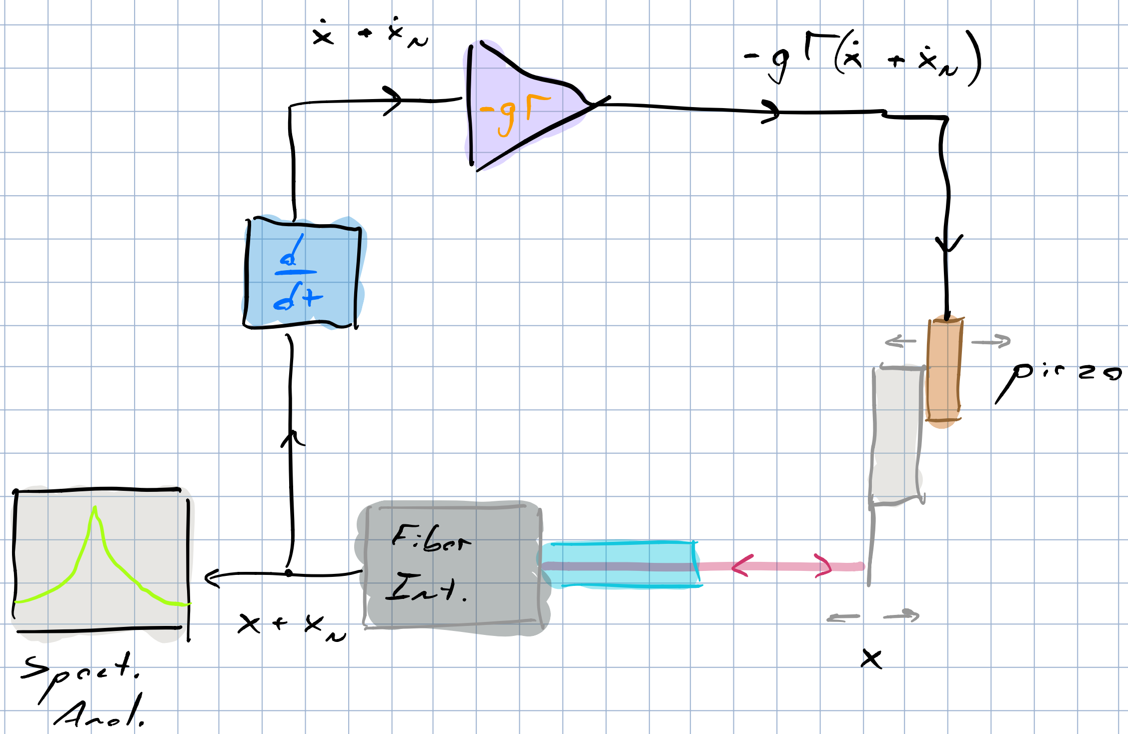
Take the equation of motion for the harmonic oscillator as our model for a mechanical mode:

← Thermal force

$$m\ddot{x} + \Gamma\dot{x} + kx = F(t)$$

$$\Gamma = \frac{m\omega_0}{Q}, \quad k = m\omega_0^2$$

Detection setup and feedback setup:



Feedback gives a modified equation of motion:

$$m\ddot{x} + \Gamma\dot{x} + k\dot{x} = F(t) - g\Gamma(\dot{x} + \dot{x}_N)$$

Let's look at one Fourier component:

$$-m\omega^2 \hat{x}(\omega) + i\omega\Gamma\hat{x}(\omega) + k\hat{x}(\omega) = \hat{f}(\omega) - ig\Gamma\omega\left[\hat{x}(\omega) + \hat{x}_n(\omega)\right]$$

$$\hat{x}(\omega) = \frac{\hat{f}(\omega) - ig\Gamma\omega\,\hat{x}_n(\omega)}{(k - m\omega^2) + i\omega\Gamma(1+g)}$$

Since $\left|\frac{a+ib}{c+id}\right|^2 = \frac{a^2+b^2}{c^2+d^2}$ and $S_x(\omega) = \lim_{T\to\infty} \frac{\hat{x}(\omega)\hat{x}^*(\omega)}{T}$:

$$\bar{S}_x(\omega) = \frac{\bar{S}_f(\omega) + g^2\Gamma^2\omega^2\bar{S}_{x_n}(\omega)}{(k - m\omega^2)^2 + \omega^2\Gamma^2(1+g)^2}$$

$$\bar{S}_x(\omega) = \left[\frac{1}{(k - m\omega^2)^2 + \omega^2\Gamma^2(1+g)^2}\right]\bar{S}_f(\omega)$$

← PSD of mode's displacement

$$+ \left[\frac{g^2\Gamma^2\omega^2}{(k - m\omega^2)^2 + \omega^2\Gamma^2(1+g)^2}\right]\bar{S}_{x_n}(\omega)$$

What do we measure?

$$\hat{x}_{meas}(\omega) = \hat{x}(\omega) + \hat{x}_n(\omega)$$

$$\hat{x}_{meas}(\omega) = \frac{\hat{f}(\omega) - i g \Gamma \omega \hat{x}_n(\omega)}{(k - m\omega^2) + i\omega\Gamma(1+g)} + \hat{x}_n(\omega)$$

$$\hat{x}_{meas}(\omega) = \frac{\hat{f}(\omega) + \left[(k - m\omega^2) + i\omega\Gamma\right]\hat{x}_n(\omega)}{(k - m\omega^2) + i\omega\Gamma(1+g)}$$

Assume the thermal force noise $\hat{f}(\omega)$ and the measured displacement noise $\hat{x}_n(\omega)$ are uncorrelated. Therefore, the noises add in quadrature:

$$\bar{S}_{meas}(\omega) = \frac{\bar{S}_f(\omega) + \left[(k - m\omega^2)^2 + \omega^2\Gamma^2\right]\bar{S}_{x_n}(\omega)}{(k - m\omega^2)^2 + \omega^2\Gamma^2(1+g)^2}$$

$$\bar{S}_{meas}(\omega) = \left[\frac{1}{(k - m\omega^2)^2 + \omega^2\Gamma^2(1+g)^2}\right]\bar{S}_f(\omega) + \left[\frac{(k - m\omega^2)^2 + \omega^2\Gamma^2}{(k - m\omega^2)^2 + \omega^2\Gamma^2(1+g)^2}\right]\bar{S}_{x_n}(\omega)$$

← PSD of measured signal

↳ Recall : $\bar{S}_F(\omega) = 4 k_B T \Gamma$

$\bar{S}_{x_n}(\omega) = \text{constant}$ ← typically from shot noise

By damping with gain g, we reduce the fluctuations in the mode of interest. This means that we cool the mode :

Recall :

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_0^\infty \bar{S}_x(\omega)\, d\omega$$

Equipartition says :

$$\frac{1}{2} k_B T_{mode} = \frac{1}{2} k \langle x^2 \rangle$$

$$T_{mode} = \frac{k}{k_B} \langle x^2 \rangle$$

$$\therefore \quad T_{mode} = \frac{k}{2\pi k_B} \int_0^\infty \bar{S}_x(\omega)\, d\omega$$

After integration ...

$$T_{mode} = \frac{T}{1+g} + \frac{k\Gamma}{4 k_B m}\left(\frac{g^2}{1+g}\right)\bar{S}_{x_N}$$

Minimize with respect to g:

$$T_{mode,min} = \sqrt{\frac{k\Gamma T}{k_B m}\bar{S}_{x_N}} = \omega_0\sqrt{\frac{\Gamma T}{k_B}\bar{S}_{x_N}}$$

This is equivalent to a minimum phonon number:

$$N_{mode,min} = \frac{k_B T_{mode,min}}{\hbar\omega_0} = \frac{1}{\hbar}\sqrt{\Gamma k_B T \bar{S}_{x_N}}$$

($\Gamma k_B T = \frac{1}{4}\bar{S}_F$)

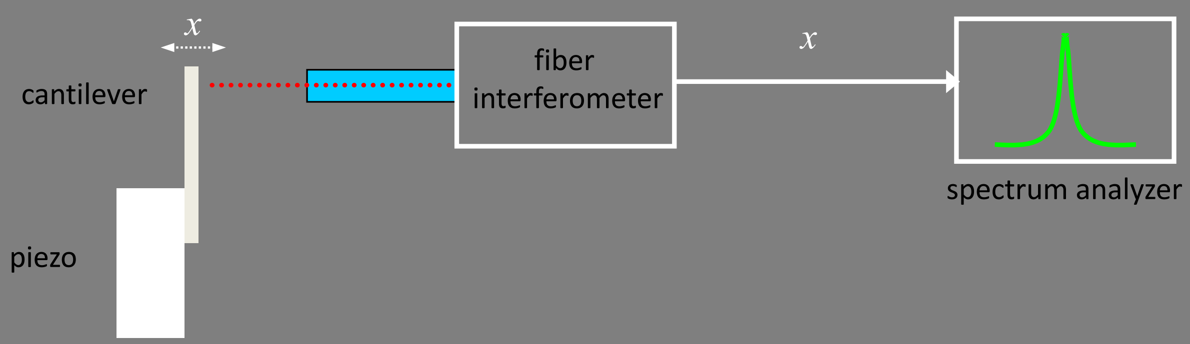
$$N_{mode,min} = \frac{1}{2\hbar}\sqrt{\bar{S}_F \bar{S}_{x_N}} = \frac{1}{\hbar}\sqrt{S_F S_{x_N}}$$

Want low T, Γ, and $\bar{S}_{x_N}$. Cold fridge, good resonators, and sensitive measurement.

Optimum is achieved, as we will discuss, when $\bar{S}_F$ is dominated by detector back-action and $\bar{S}_{x_N}$ is quantum limited. More on this to come...

Usual Cantilever Motion Detection

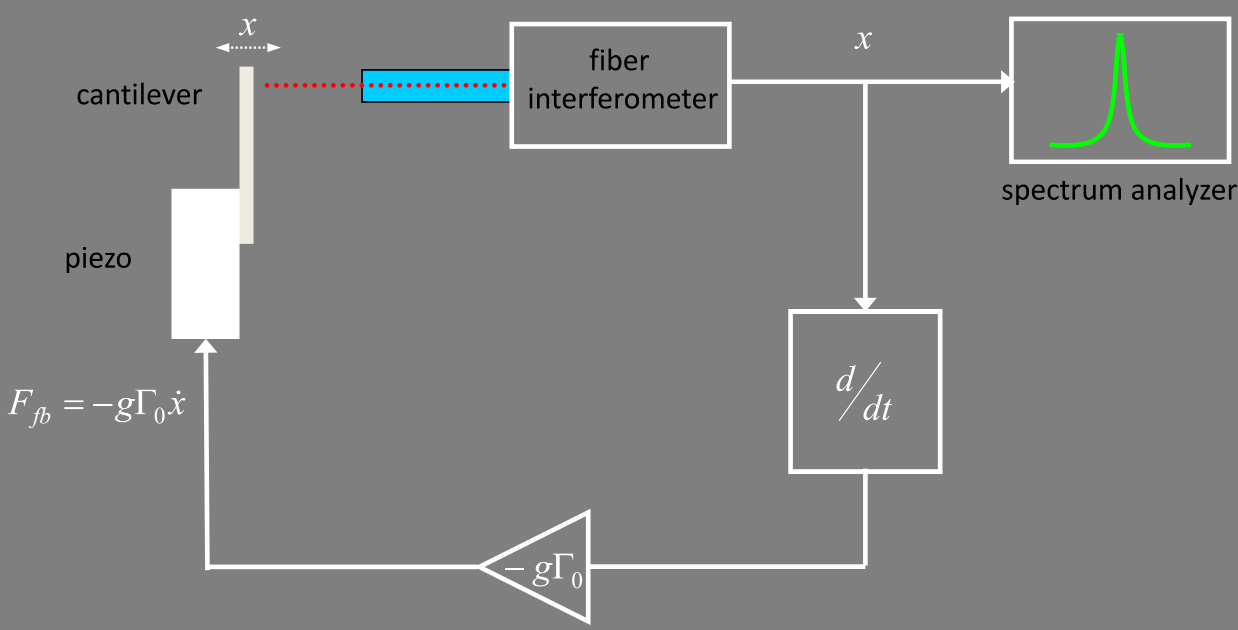
$$m\ddot{x} + \Gamma_0 \dot{x} + kx = F_{th}$$



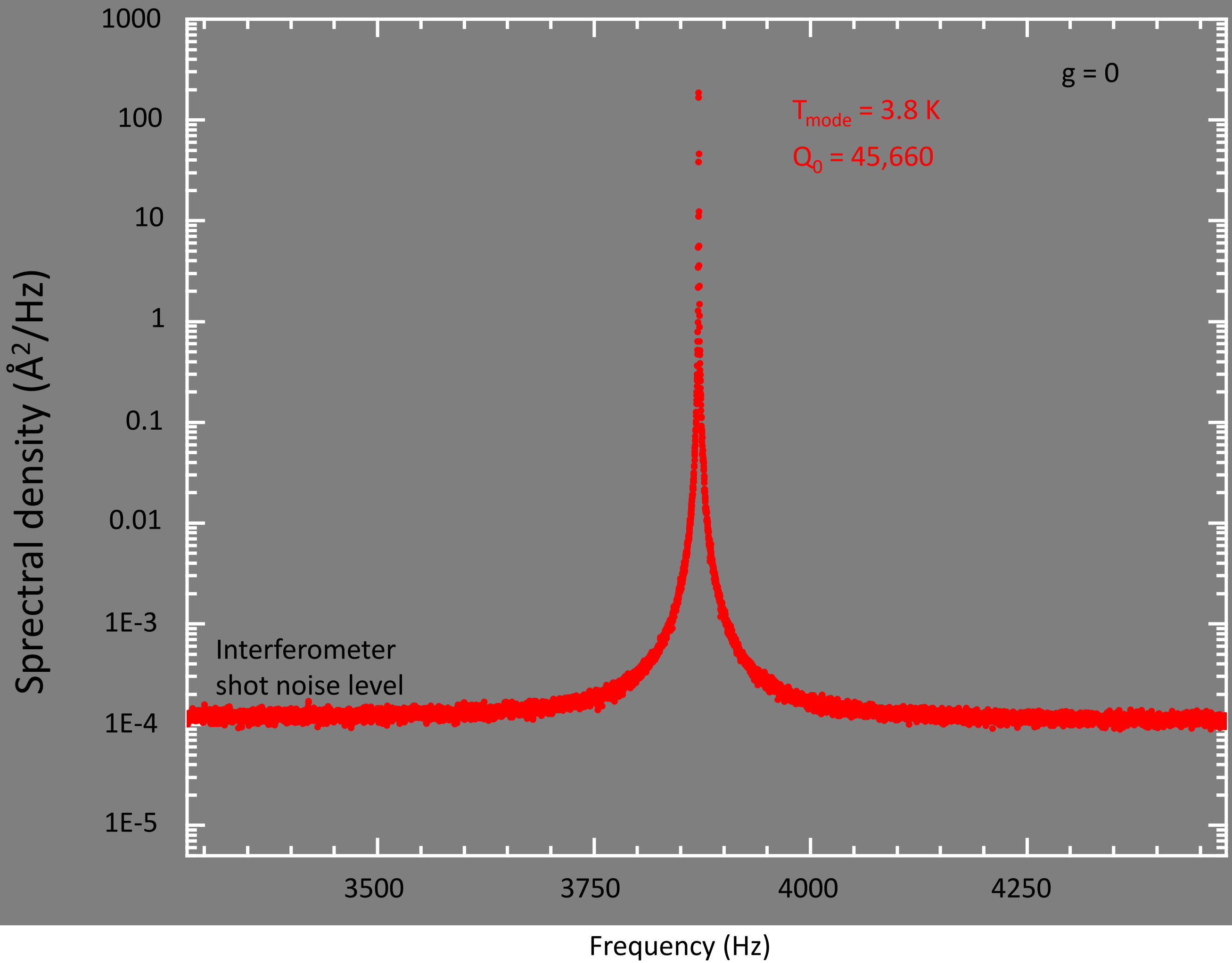
Simple Electronic Damping

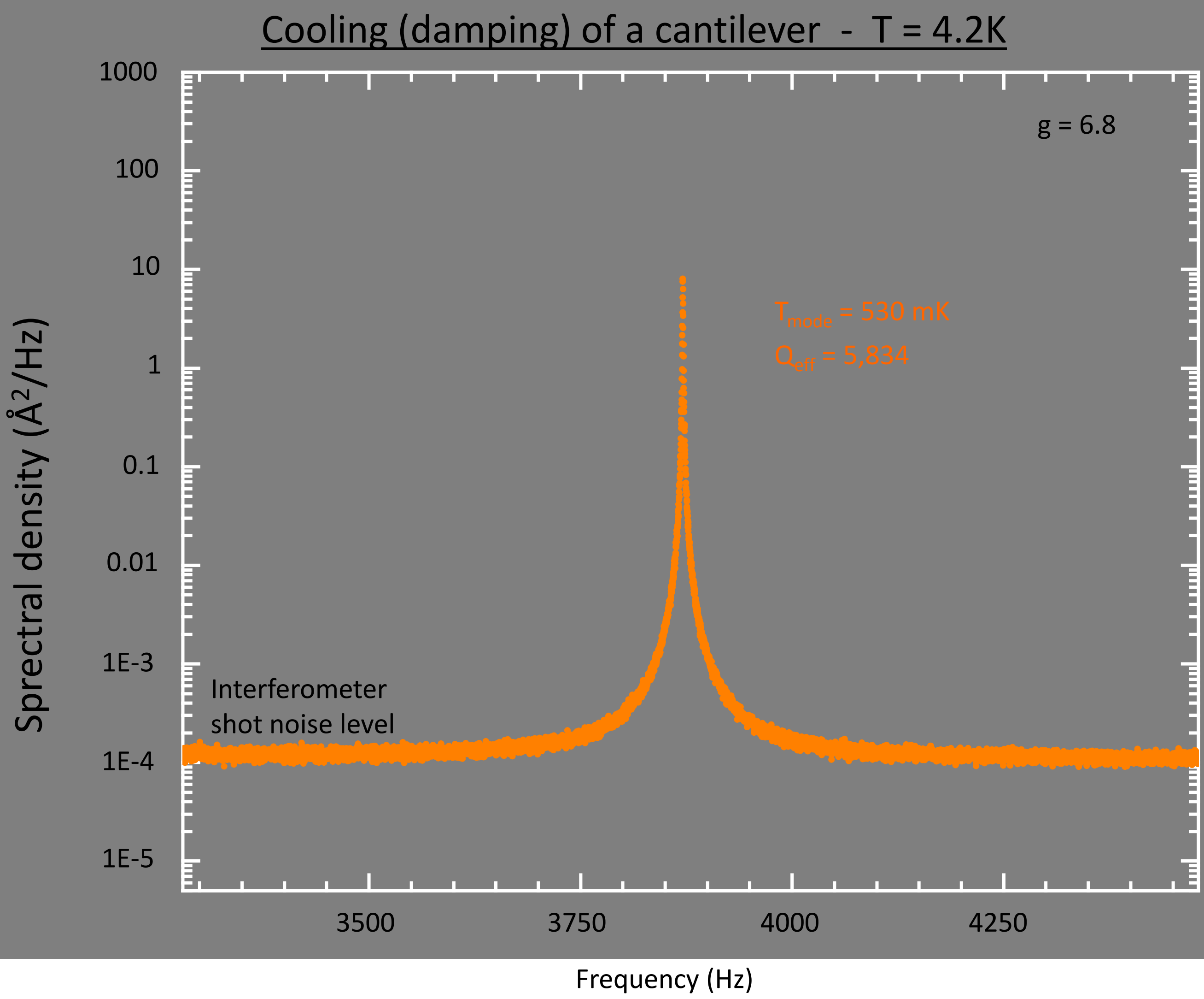
$$m\ddot{x} + \Gamma_0\dot{x} + kx = F_{th} - g\Gamma_0\dot{x}$$

damping

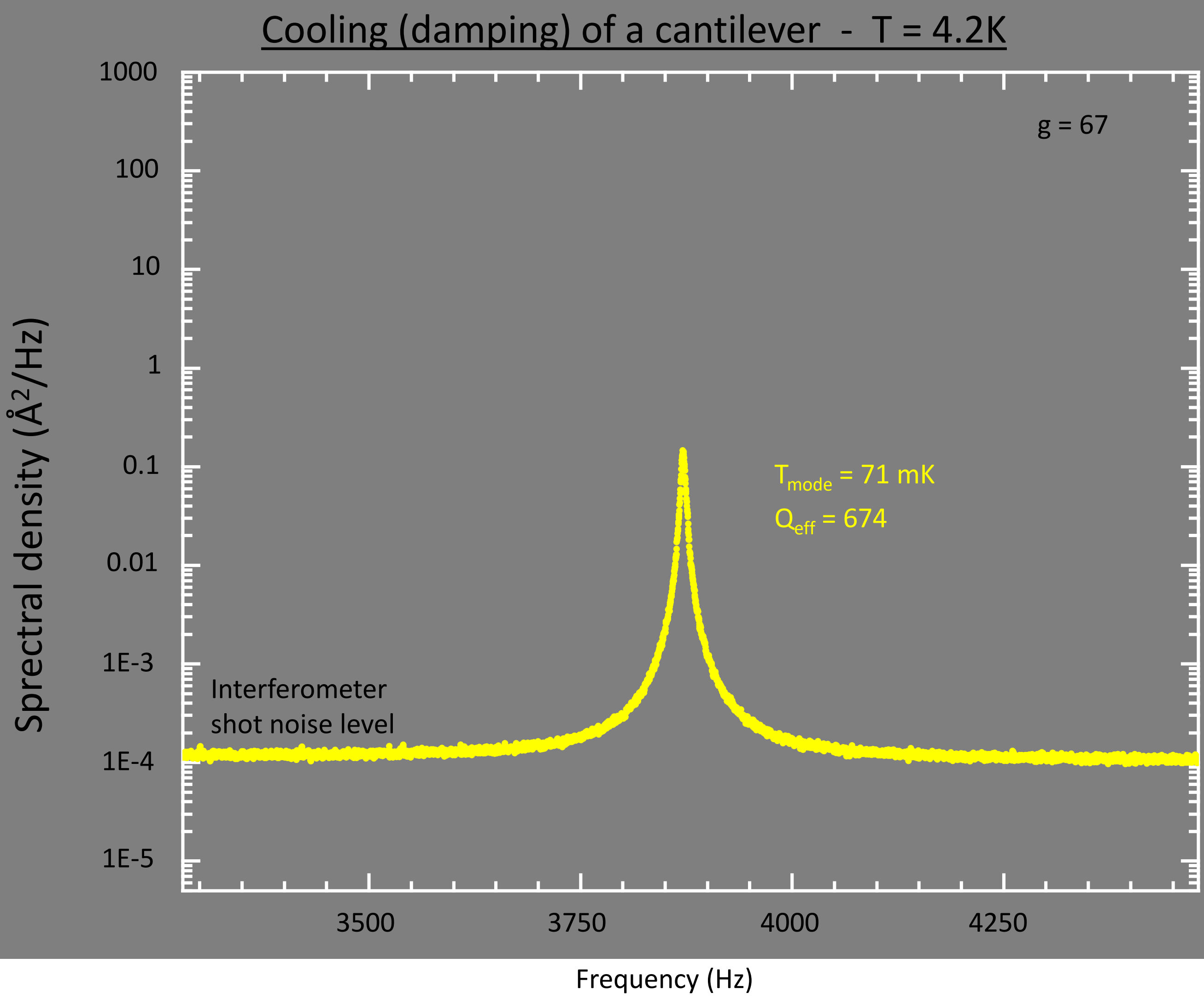


Cooling (damping) of a cantilever - T = 4.2K



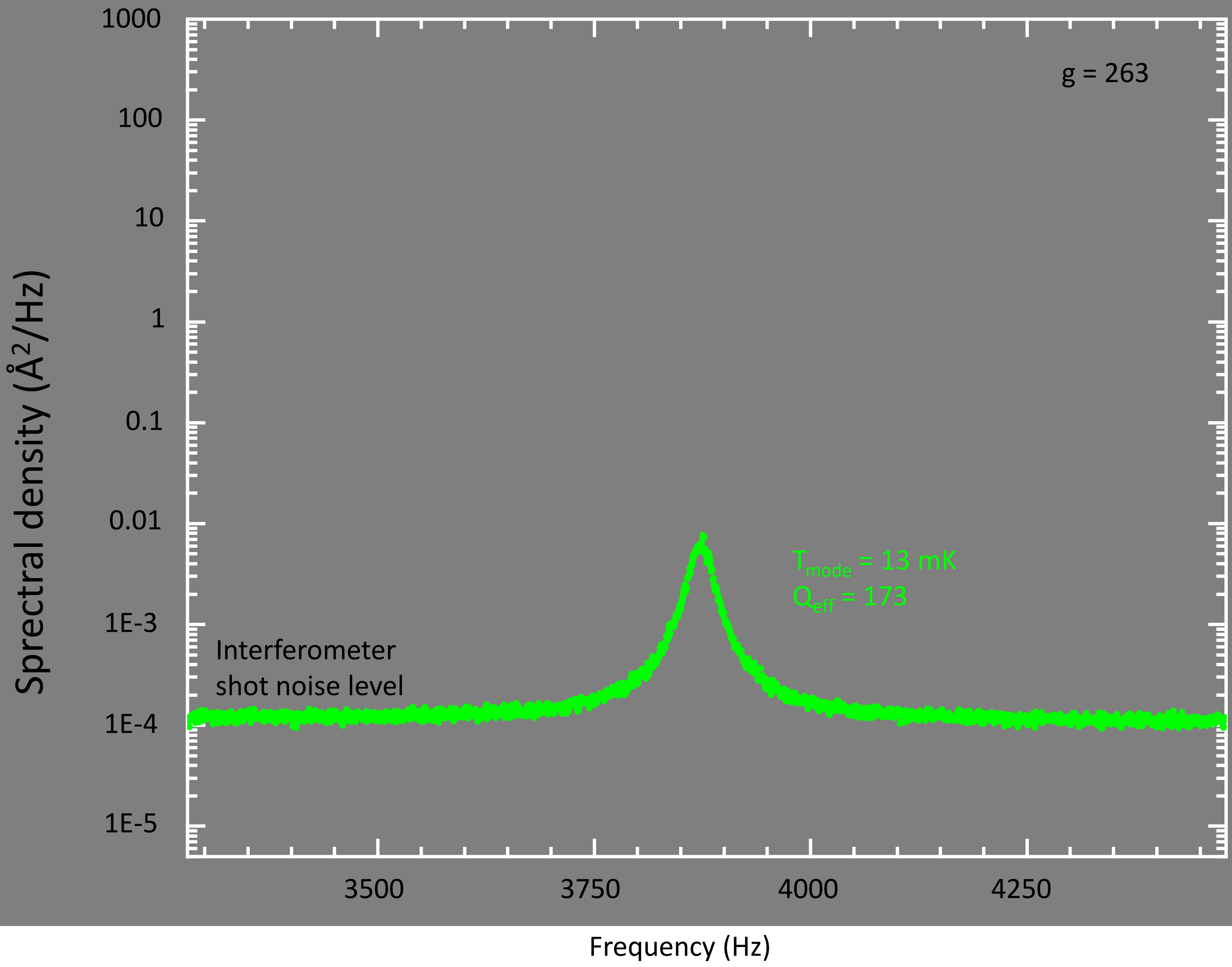


Cooling (damping) of a cantilever - T = 4.2K
g = 6.8
T_{mode} = 530 mK
Q_{eff} = 5,834
Interferometer
shot noise level
1000
100
10
1
0.1
0.01
1E-3
1E-4
1E-5
3500
3750
4000
4250
Sprectral density (Å^2/Hz)
Frequency (Hz)

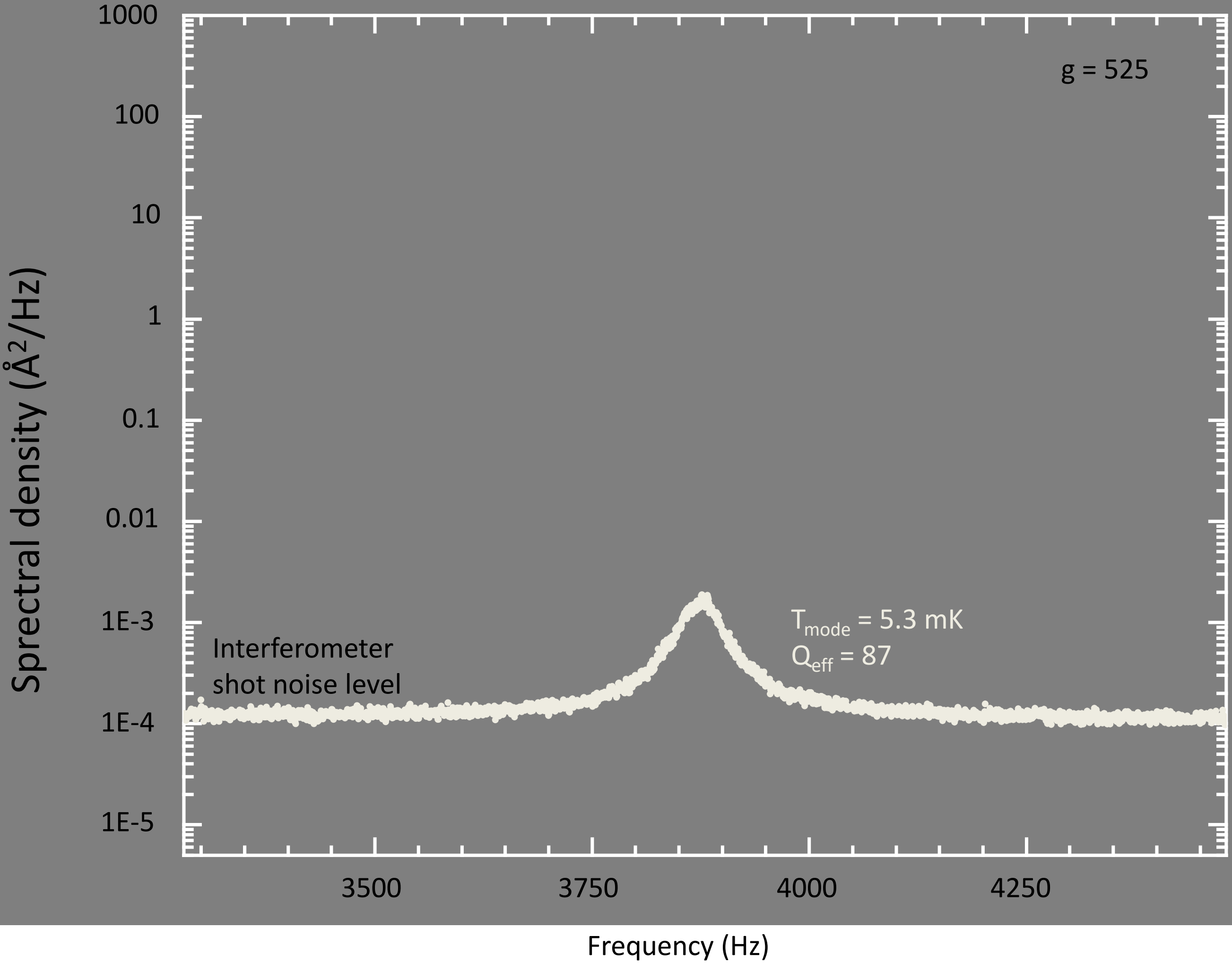


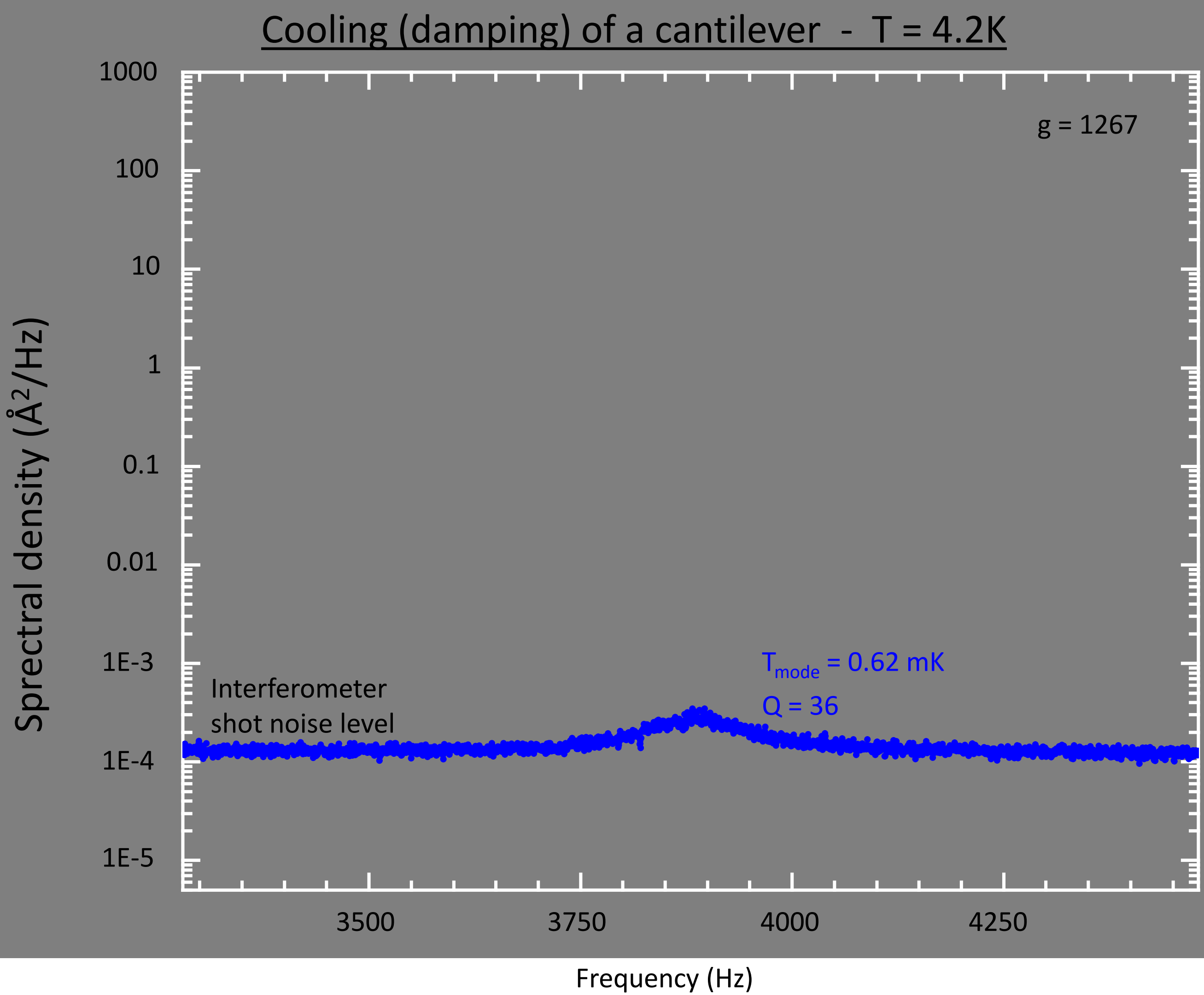
Cooling (damping) of a cantilever - T = 4.2K
g = 67
T_{mode} = 71 mK
Q_{eff} = 674
Interferometer
shot noise level
1000
100
10
1
0.1
0.01
1E-3
1E-4
1E-5
3500
3750
4000
4250
Sprectral density (Å^2/Hz)
Frequency (Hz)

Cooling (damping) of a cantilever - T = 4.2K

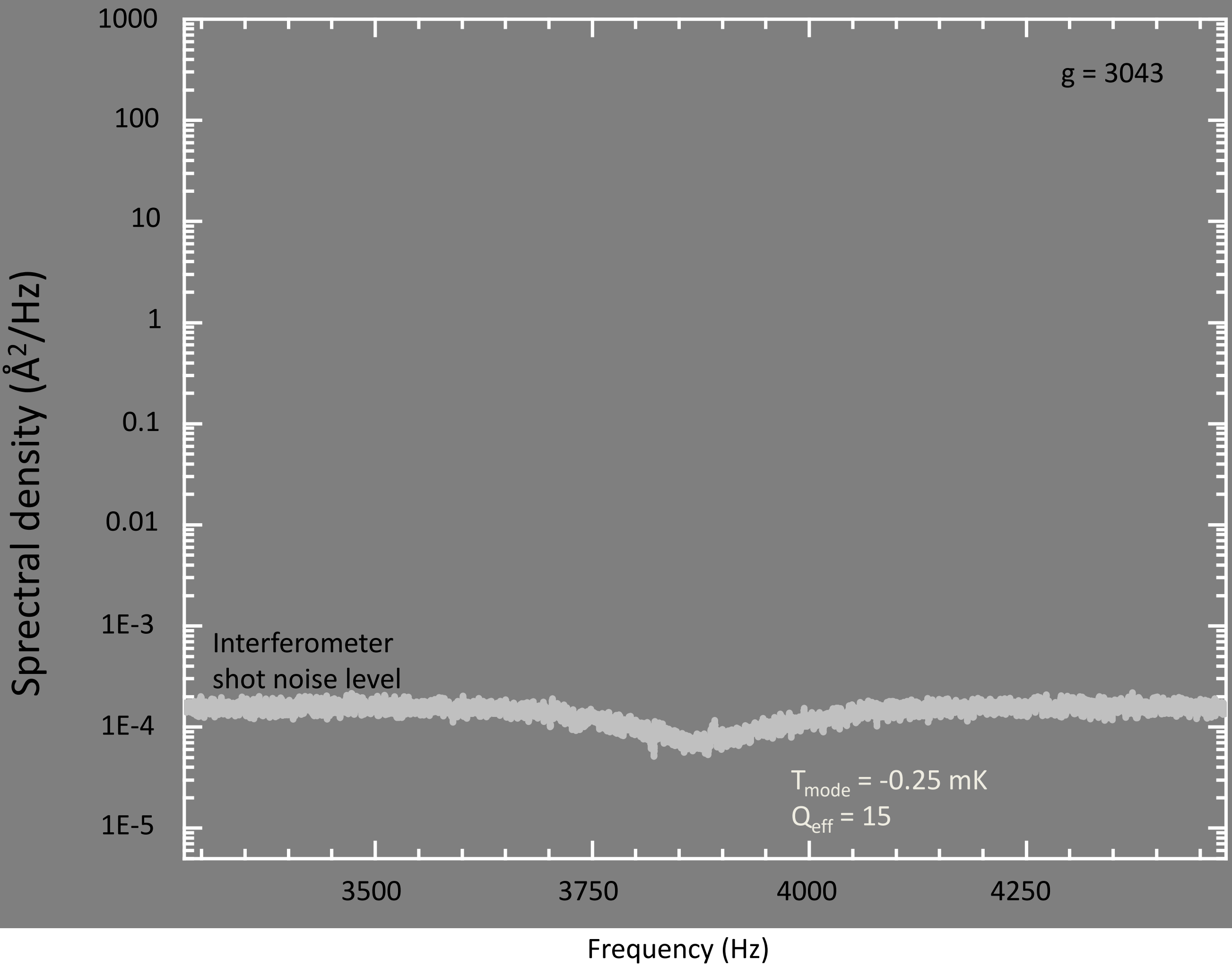


Cooling (damping) of a cantilever - T = 4.2K

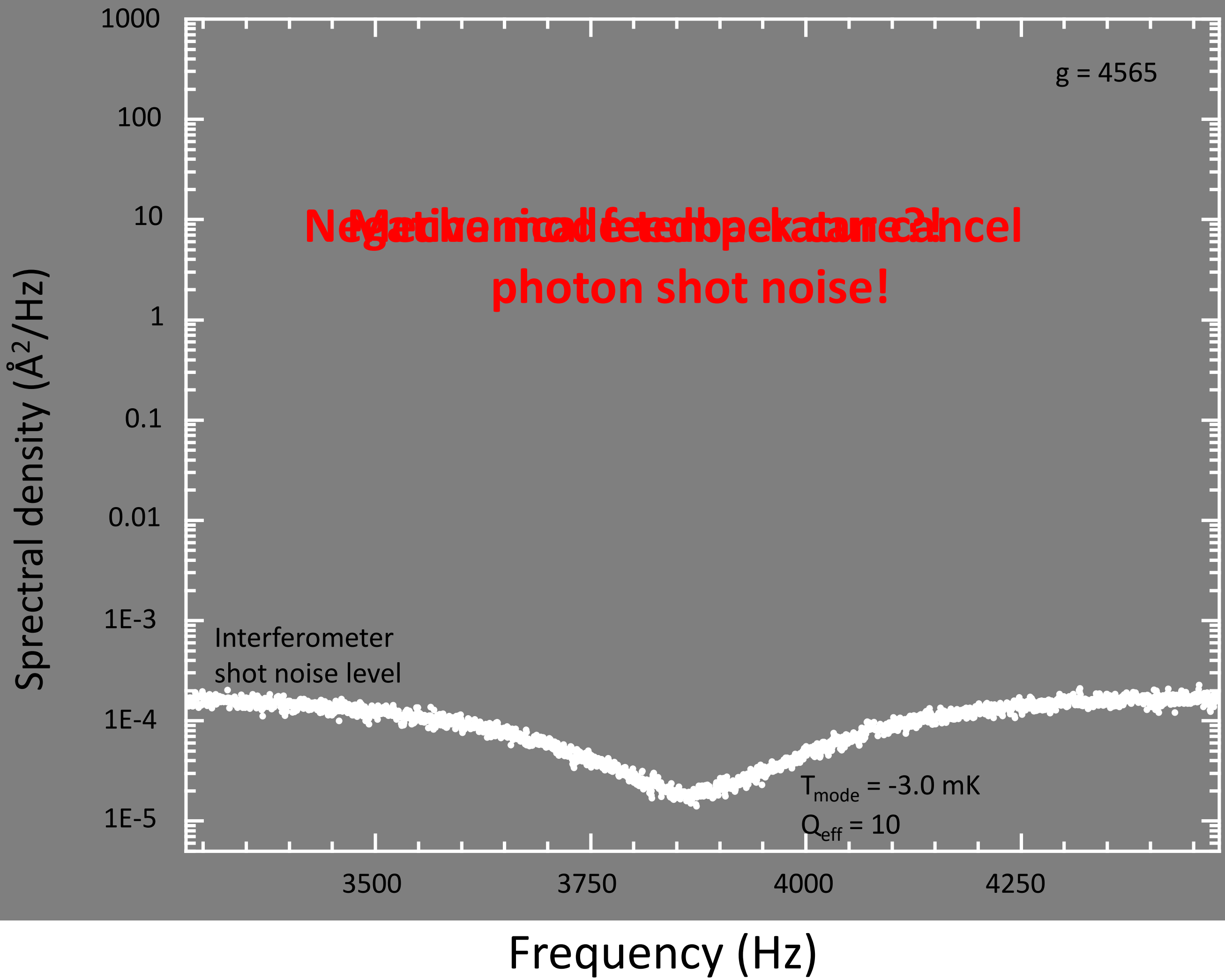



Cooling (damping) of a cantilever - T = 4.2K
g = 1267
T_{mode} = 0.62 mK
Q = 36
Interferometer
shot noise level
1000
100
10
1
0.1
0.01
1E-3
1E-4
1E-5
3500
3750
4000
4250
Sprectral density (Å^2/Hz)
Frequency (Hz)

Cooling (damping) of a cantilever - T = 4.2K



Cooling (damping) of a cantilever - T = 4.2K

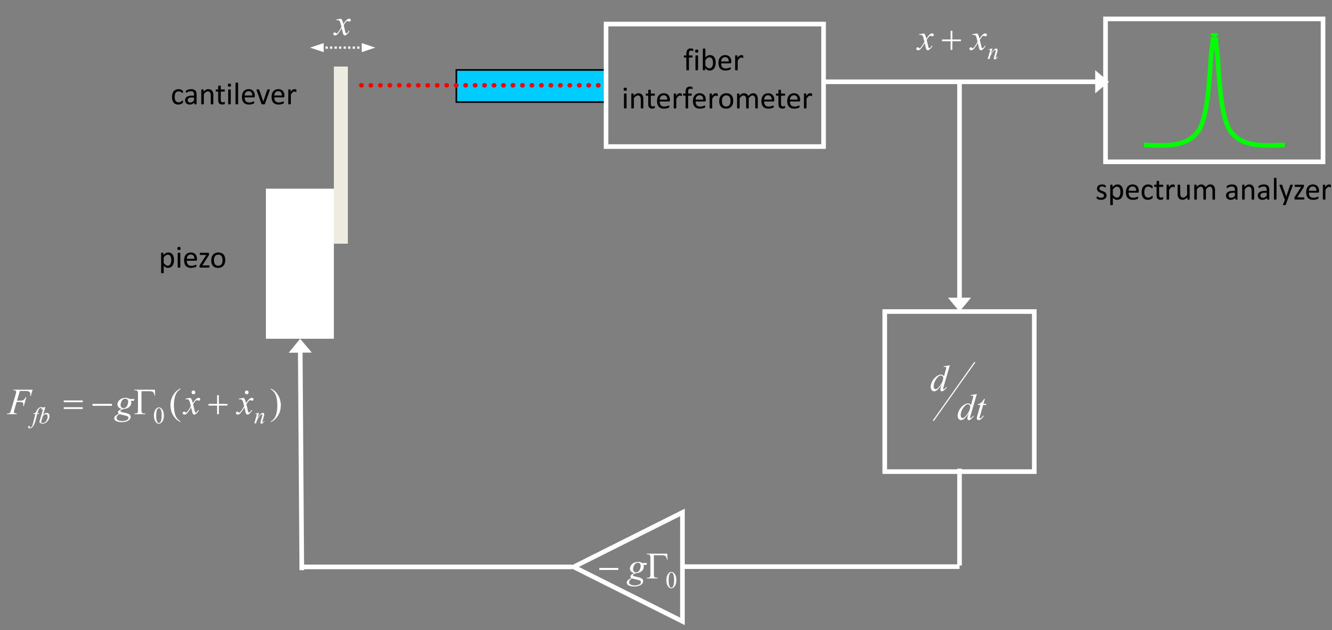


Experimental setup

damping

$$m\ddot{x} + \Gamma_0 \dot{x} + kx = F_{th} - g\Gamma_0(\dot{x} + \dot{x}_n)$$

measurement noise



Cantilever Noise Temperature with Feedback

$$m\ddot{x} + \Gamma_0 \dot{x} + kx = F_{th} - g\Gamma_0\left(\dot{x} + \dot{x}_n\right)$$

Effective Q with feedback:

$$Q_{eff} = \frac{Q_0}{1+g} = \frac{k}{\omega_c\left(1+g\right)\Gamma_0}$$

Measured spectral density:

$$S_{x+x_n}(\omega) = \left[\frac{\omega_c^4 / k^2}{\left(\omega^2 - \omega_c^2\right)^2 + \omega_c^2\omega^2 / Q_{eff}^2}\right] S_{F_{th}} + \left[\frac{\left(\omega^2 - \omega_c^2\right)^2 + \omega_c^2\omega^2 / Q_0^2}{\left(\omega^2 - \omega_c^2\right)^2 + \omega_c^2\omega^2 / Q_{eff}^2}\right] S_{x_n}$$

Actual cantilever spectral density:

$$S_x(\omega) = \frac{\omega_c^4 / k^2}{\left(\omega^2 - \omega_c^2\right)^2 + \omega_c^2\omega^2 / Q_{eff}^2}\left[S_{F_{th}} + \frac{g^2 k^2 \omega^2}{\omega_c^2 Q_0^2} S_{x_n}\right]$$

Cantilever mode temperature:

$$T_{\mathrm{mode}} = \frac{k\left\langle x^2 \right\rangle}{k_B}$$

$$T_{\mathrm{mode}} = \frac{T}{1+g} + \frac{1}{4}\frac{k}{k_B}\omega_c\left(\frac{g^2}{1+g}\right)\frac{1}{Q_0} S_{x_n}$$

Cantilever Noise Temperature with Feedback

$$m\ddot{x} + \Gamma_0 \dot{x} + kx = F_{th} - g\Gamma_0 \left(\dot{x} + \dot{x}_n \right)$$

Effective Q with feedback:

$$Q_{eff} = \frac{Q_0}{1+g} = \frac{k}{\omega_c \left(1+g\right)\Gamma_0}$$

Measured spectral density:

$$S_{x+x_n}(\omega) = \left[\frac{\omega_c^4 / k^2}{\left(\omega^2 - \omega_c^2\right)^2 + \omega_c^2 \omega^2 / Q_{eff}^2} \right] S_{F_{th}} + \left[\frac{\left(\omega^2 - \omega_c^2\right)^2 + \omega_c^2 \omega^2 / Q_0^2}{\left(\omega^2 - \omega_c^2\right)^2 + \omega_c^2 \omega^2 / Q_{eff}^2} \right] S_{x_n}$$

Actual cantilever spectral density:

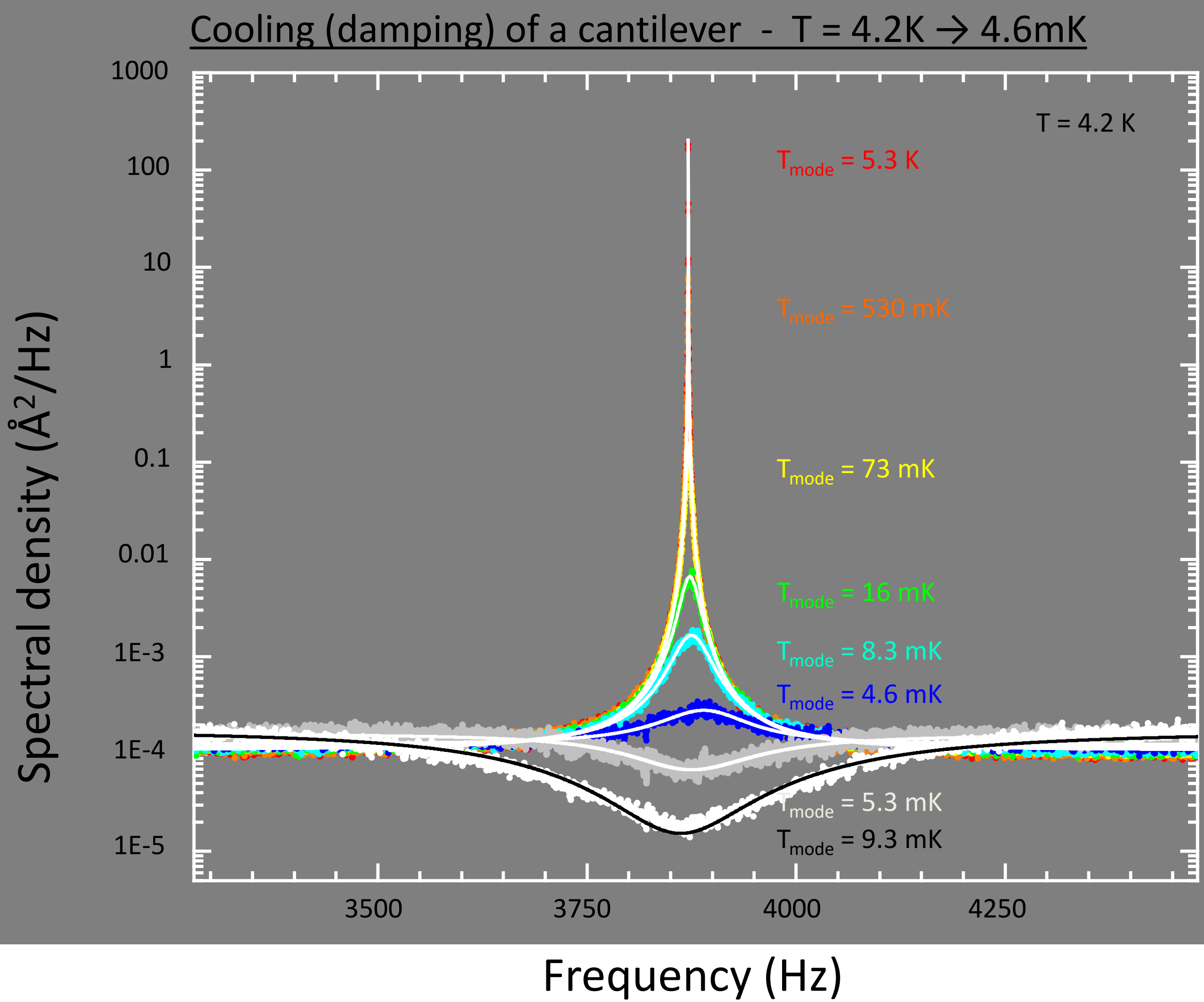
$$S_x(\omega) = \frac{\omega_c^4 / k^2}{\left(\omega^2 - \omega_c^2\right)^2 + \omega_c^2 \omega^2 / Q_{eff}^2} \left[S_{F_{th}} + \frac{g^2 k^2 \omega^2}{\omega_c^2 Q_0^2} S_{x_n} \right]$$

Cantilever mode temperature:

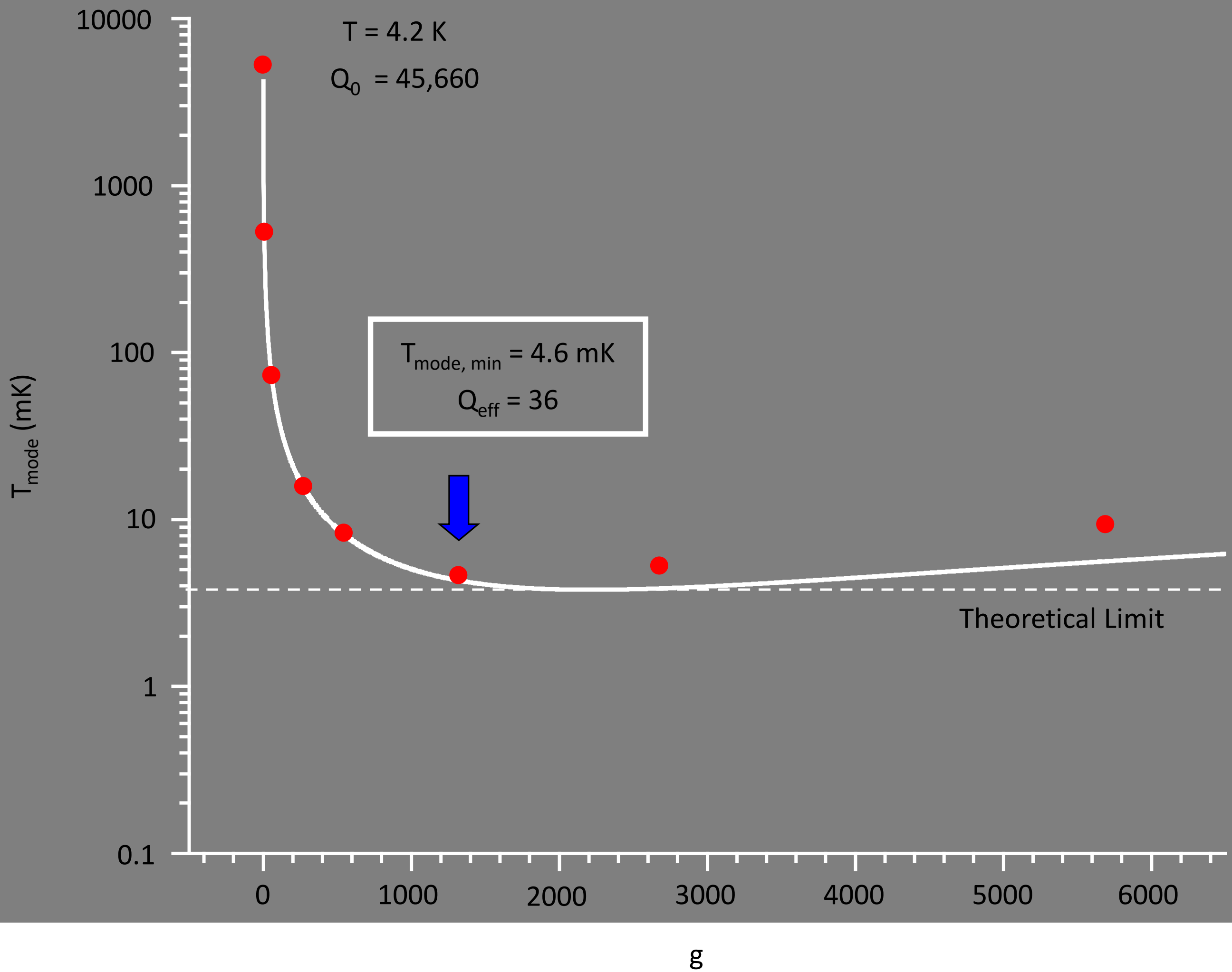
$$T_{mode} = \frac{k \left\langle x^2 \right\rangle}{k_B}$$

For optimum feedback gain

$$T_{\mathrm{mode,min}} = \sqrt{\frac{k \omega_c T}{k_B Q_0} S_{x_n}}$$


Cooling (damping) of a cantilever - T = 4.2K → 4.6mK
1000
100
10
1
0.1
0.01
1E-3
1E-4
1E-5
Spectral density (Å2/Hz)
3500
3750
4000
4250
Frequency (Hz)
T = 4.2 K
Tmode = 5.3 K
Tmode = 530 mK
Tmode = 73 mK
Tmode = 16 mK
Tmode = 8.3 mK
Tmode = 4.6 mK
Tmode = 5.3 mK
Tmode = 9.3 mK

Cooling (damping) of a cantilever – model and experiment



Cooling (damping) of a cantilever – model and experiment