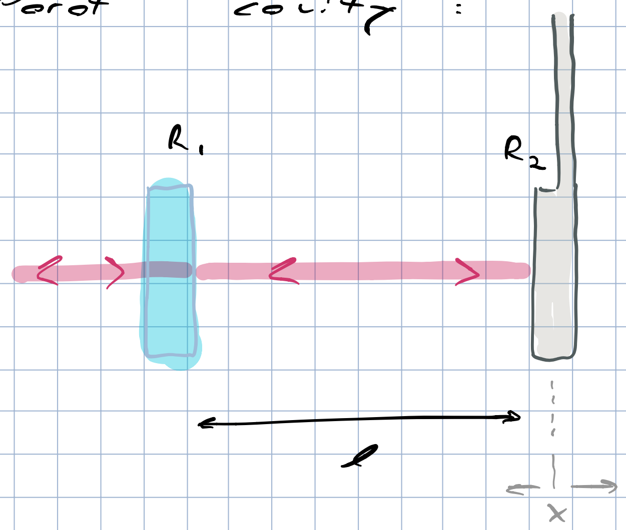
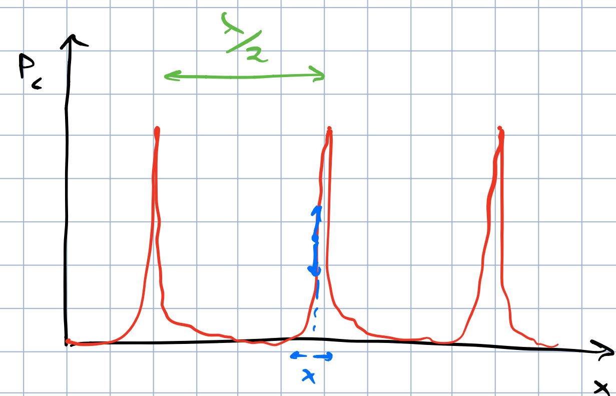
Lecture 9 (11. 11. 2020)

Cavity Cooling

We can cool our mechanical mode using an optical cavity. Recall our Fabry-Perot cavity:





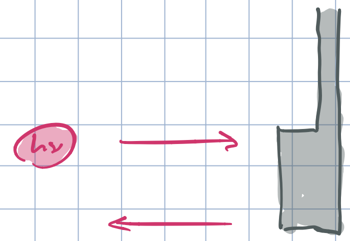
Power in cavity

$$P_c = P_I \left[\frac{\frac{(1 - R_1)}{(1 - \sqrt{R_1 R_2})^2}}{1 + F \sin^2\left(\frac{2\pi l}{\lambda}\right)} \right]$$

where $F = \dfrac{4\sqrt{R_1 R_2}}{(1 - \sqrt{R_1 R_2})^2}$

Optical forces act on the compliant mirror. Most prominently, these are photo-thermal and radiation pressure. Both forces act with a characteristic time delay. We will now consider only the radiation pressure force.

I. Semi-classical Picture



n : photons arriving per unit time

Momentum kick to lever for each photon : $\Delta p = 2 \frac{h\nu}{c}$

$$\therefore \quad \frac{2 n \Delta t \, h\nu}{c} = \Delta p$$

← momentum in Δt with photon flux n.

$$\frac{\Delta p}{\Delta t} = \frac{2 n h \nu}{c}$$

radiation force

$$\mathcal{F} = \frac{2P}{c}$$

← power (intensity)

The force $\mathcal{F}$ is not instantaneous. The finesse F introduces a time lag in the radiation force. We can therefore write that the actual radiation force F_{opt} lags $\mathcal{F}$ a little bit :

$$\dot{F}_{opt}(t) = \frac{\mathcal{F}(x) - F_{opt}(t)}{\tau}$$

where τ is a time delay introduced by the optical cavity.

$$\tau \dot{F}_{opt}(t) + F_{opt}(t) = \mathcal{F}(x)$$

Let's focus on the Fourier component at ω :

$$i\omega\tau \hat{F}_{opt}(\omega) + \hat{F}_{opt}(\omega) = \hat{\mathcal{F}}(\omega)$$

$$\hat{F}_{opt}(\omega) = \frac{\hat{\mathcal{F}}(\omega)}{1 + i\omega\tau}$$

Linearizing with respect to small displacements x from the equilibrium position x_0 :

$$\hat{F}_{opt}(\omega) = \frac{\mathcal{F}'(x_0)\,\hat{x}(\omega)}{1 + i\omega\tau}$$

If we now put this optical force into our equation for the harmonic oscillator, we have :

$$m\ddot{x} + \Gamma\dot{x} + k\dot{x} = f(t) + F_{opt}$$

For one Fourier component at ω :

$$-m\omega^2\hat{x}(\omega) + i\omega\Gamma\hat{x}(\omega) + k\hat{x}(\omega) = \hat{f}(\omega) + \underbrace{\frac{\mathcal{F}'(x_0)\,\hat{x}(\omega)}{1 + i\omega\tau}}_{\frac{\mathcal{F}'(x_0)\,\hat{x}(\omega)}{1+\omega^2\tau^2} - i\frac{\omega\tau\,\mathcal{F}'(x_0)\,\hat{x}(\omega)}{1+\omega^2\tau^2}}$$

$$-m\omega^2 \hat{x}(\omega) + i\omega \underbrace{\boxed{\left[\Gamma + \frac{\tau \mathcal{F}'(x_0)}{1+\omega^2\tau^2}\right]}}_{\text{change in damping (Q)}} \hat{x}(\omega) + \underbrace{\boxed{\left[k - \frac{\mathcal{F}'(x_0)}{1+\omega^2\tau^2}\right]}}_{\text{change in spring const. (freq.)}} \hat{x}(\omega) = \hat{F}(\omega)$$

Optical spring : $\dfrac{-\mathcal{F}'(x_0)}{1+\omega^2\tau^2} = k_{opt}$

Optical damping : $\dfrac{\tau \mathcal{F}'(x_0)}{1+\omega^2\tau^2} = \Gamma_{opt}$

We can now go through the same analysis as with the feedback cooling. We simply use a renormalized spring constant k', which includes the optical spring effect and we use $g = \dfrac{1}{\Gamma} \dfrac{\tau \mathcal{F}'(x_0)}{1+\omega^2\tau^2}$.

The analysis follows exactly the analysis for feedback cooling. In the limit where the measurement noise x_N is negligible :

$$T_{mode} \approx \frac{T}{1+g}$$

$$T_{mode} \approx T \frac{1}{1 + \frac{1}{\Gamma} \underbrace{\frac{\tau \mathcal{F}'(x_0)}{1 + \omega^2 \tau^2}}_{\Gamma_{opt}}}$$

$$\boxed{T_{mode} \approx T \left(\frac{\Gamma}{\Gamma + \Gamma_{opt}} \right)}$$

II. Quantum Picture

In order to discuss cavity cooling – and any other cooling for that matter – which reaches $k_B T \sim \hbar\omega_0$, we have to introduce a quantum mechanical treatment. First of all, we have to define a quantum analoge to the classical spectral density, i.e. the quantum spectral density. For example for quantum noise on the displacement of an oscillator, given by the quantum operator $\hat{x}$, we define:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \langle \hat{x}(t)\, \hat{x}(0) \rangle \, e^{i\omega t} \, dt$$

double subscript indicates quantum nature

Classical analoge:

$$S_x(\omega) = \int_{-\infty}^{\infty} \langle x(t)\, x(0) \rangle \, e^{i\omega t} \, dt$$

Classical

Here, $\hat{x}$ is a quantum operator and $\langle \ \rangle$ is a quantum statistical average using a density matrix.

Let's first consider a simple harmonic oscillator with no dissipation. The Hamiltonian for this system is:

momentum

angular resonance freq.

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2}$$

position

mass

Using the Heisenberg picture we have for any operator $\hat{A}$:

$$\frac{d\hat{A}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}] + \frac{\partial \hat{A}}{\partial t}$$

Recall: $[\hat{x}, \hat{p}] = i\hbar$

$$\therefore \quad \frac{d\hat{x}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{x}] = \frac{\hat{p}}{m}$$

$$\frac{d\hat{p}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{p}] = -m\omega_0^2 \hat{x}$$

$$\frac{d^2\hat{x}}{dt^2} + \omega_0^2 \hat{x} = 0$$

$$\frac{d^2\hat{p}}{dt^2} + \omega_0^2 \hat{p} = 0$$

Solutions take the form:

$$\hat{x}(t) = \hat{x}(0)\cos(\omega_0 t) + \frac{\hat{p}(0)}{m}\sin(\omega_0 t)$$
$$\hat{p}(t) = \hat{p}(0)\cos(\omega_0 t) - m\omega_0 \hat{x}(0)\sin(\omega_0 t)$$

The correlation function thus looks like:

$$\langle \hat{x}(t)\,\hat{x}(0)\rangle = \langle \hat{x}(0)\,\hat{x}(0)\rangle \cos(\omega_0 t) + \langle \hat{p}(0)\,\hat{x}(0)\rangle \frac{1}{m\omega_0}\sin(\omega_0 t)$$

Let's now introduce the creation and annihilation operators $\hat{a}^+$ and $\hat{a}$:

$$\hat{x} = x_{ZPF}(\hat{a}^+ + \hat{a})$$
$$\hat{p} = \frac{i\hbar}{2x_{ZPF}}(\hat{a}^+ - \hat{a})$$

where $x_{ZPF}^2 = \langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega_0}$

and $[\hat{a}, \hat{a}^+] = 1$, $\hat{n} = \hat{a}^+\hat{a}$

In thermal equilibrium, we have

$$\langle \hat{x}\,\hat{p}\rangle = \left\langle \frac{i\hbar}{2}(\hat{a}^+\hat{a}^+ - \hat{a}^+\hat{a} + \hat{a}\hat{a}^+ - \hat{a}\hat{a})\right\rangle = \frac{i\hbar}{2}$$

$$\longrightarrow \langle \hat{p}\,\hat{x}\rangle = \left\langle \frac{i\hbar}{2}(\hat{a}^+\hat{a}^+ + \hat{a}^+\hat{a} - \hat{a}\hat{a}^+ - \hat{a}\hat{a})\right\rangle = -\frac{i\hbar}{2}$$

$$\langle \hat{x}\,\hat{x}\rangle = \langle x_{ZPF}^2(\hat{a}^+\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+ + \hat{a}\hat{a})\rangle$$

$$\longrightarrow \langle \hat{x}\hat{x}\rangle = \langle x_{ZPF}^2(2\hat{n} + 1)\rangle$$

As a result:

$$\langle \hat{x}(t)\,\hat{x}(0)\rangle = \langle \hat{x}(0)\,\hat{x}(0)\rangle \frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}$$

$$+ \langle \hat{p}(0)\,\hat{x}(0)\rangle \frac{1}{m\omega_0} \frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i}$$

$$\langle \hat{x}(t)\,\hat{x}(0)\rangle = x_{ZPF}^2 \left[\langle 2\hat{n}+1\rangle \frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2} - \langle 1\rangle \frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2} \right]$$

$$\langle \hat{x}(t)\,\hat{x}(0)\rangle = x_{ZPF}^2 \left[\langle \hat{n}\rangle e^{i\omega_0 t} + \langle \hat{n}+1\rangle e^{-i\omega_0 t} \right]$$

The quantum spectral density for the harmonic oscillator is then:

(1) $$\boxed{S_{xx}(\omega) = 2\pi\, x_{ZPF}^2 \left[\langle \hat{n}\rangle \delta(\omega+\omega_0) + \langle \hat{n}+1\rangle \delta(\omega-\omega_0) \right]}$$

oscillator emits energy

oscillator absorbs energy

Note: Quantum spectral density is asymmetric! This is because the autocorrelation function is complex, since $\hat{x}$ does not commute with itself at different times.

In the high temperature limit, $k_B T >> \hbar\omega_0$, we retrieve a symmetric and purely classical spectral density :

For $k_B T >> \hbar\omega_0$,

$$\langle \hat{n} \rangle \sim \langle \hat{n} + 1 \rangle \sim \frac{k_B T}{\hbar\omega_0}$$

$$\therefore \lim_{k_B T >> \hbar\omega_0} S_{xx}(\omega) = \pi \frac{k_B T}{m\omega_0^2}\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0) \right] = S_x(\omega)$$

classical

In the classical limit, we should retrieve the equipartition theorem ...

$$\langle x^2 \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\lim_{k_B T >> \hbar\omega_0} S_{xx}(\omega) \right) d\omega = \frac{1}{2}\frac{k_B T}{m\omega_0^2}(2)$$

$$\langle x^2 \rangle = \frac{k_B T}{m\omega_0^2}$$

$$\frac{1}{2} m\omega_0^2 \langle x^2 \rangle = \frac{1}{2} k_B T$$ ✓

In the low temperature limit, $k_B T << \hbar\omega_0$.
In that case :

$$\langle \hat{n} \rangle \sim 0 \; ; \; \langle \hat{n} + 1 \rangle \sim 1$$

$$\therefore \quad \lim_{k_B T \ll \hbar\omega_0} S_{xx}(\omega) = 2\pi \, x_{zpf}^2 \, \delta(\omega - \omega_0)$$

Deep in the quantum regime :

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\lim_{k_B T \ll \hbar\omega_0} S_{xx}(\omega) \right) d\omega = x_{zPF}^2$$

$$\langle x^2 \rangle = x_{zPF}^2$$