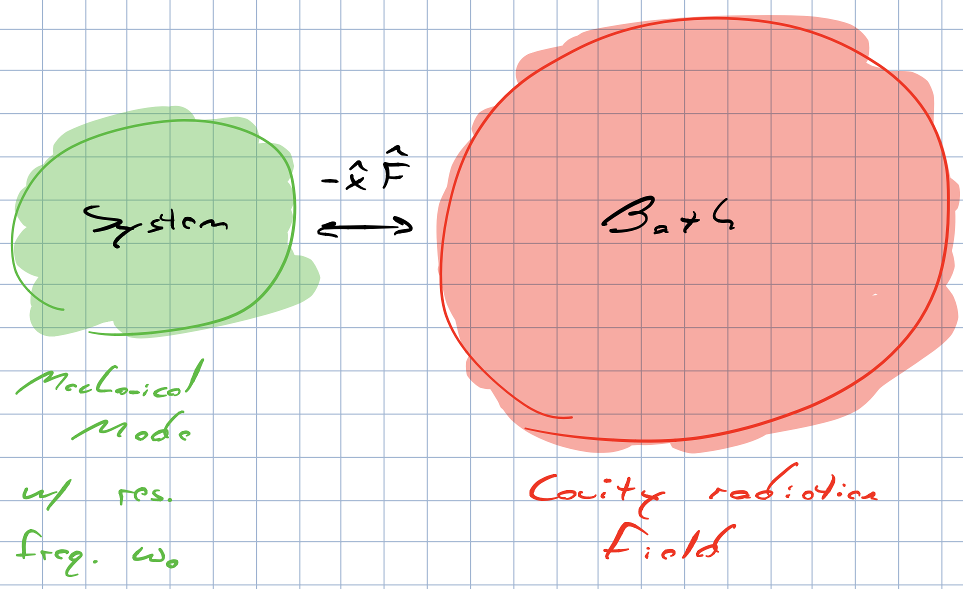
Lecture 10 (18.11.2020)

Cavity Cooling

II. Quantum Picture

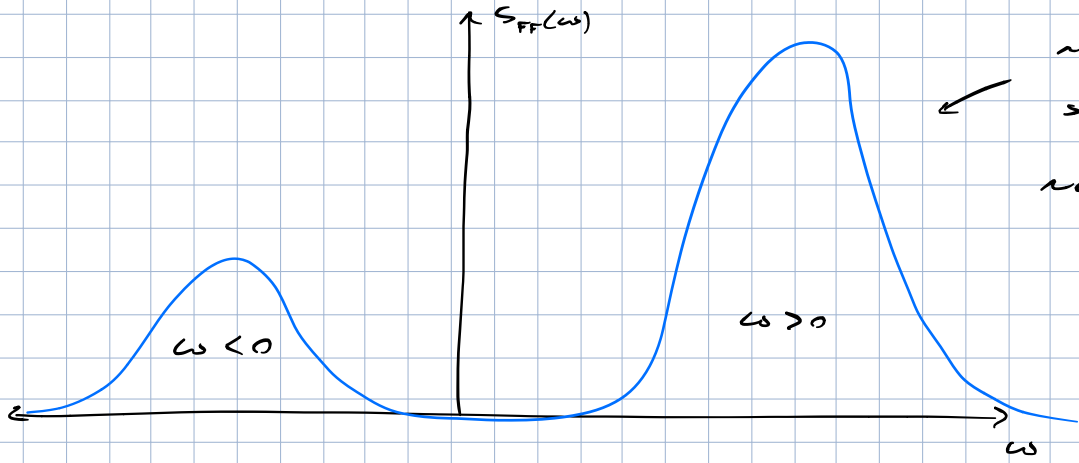
Now that we've seen how quantum spectral densities look, let's get back to a quantum treatment of cavity cooling.

Let's suppose that our mechanical mode (harmonic oscillator) is coupled to the radiation field of a cavity:



The coupling goes through a radiation pressure force $\hat{F}$, whose spectral density is:

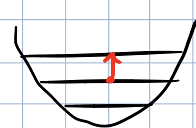
$$S_{FF}(\omega) = \int_{-\infty}^{\infty} \langle \hat{F}(t)\,\hat{F}(0)\rangle\, e^{i\omega t}\, dt$$



not necessarily symmetric noise spectrum

Noise in $\hat{F}$ at the oscillator frequency ω_0 can cause transitions between its harmonic oscillator eigenstates. If F is small and the noise has a short correlation time, we can use perturbation theory to derive the transition rates. One finds that the transition rate from $|n\rangle$ to $|n+1\rangle$ is:

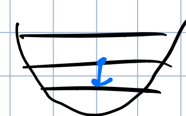
$$\Gamma_{n \to n+1} = \frac{x_{ZPF}^2}{\hbar^2} (n+1) \, S_{FF}(-\omega_0)$$



$\propto S_{FF}(-\omega_0)$

and from $|n\rangle$ to $|n-1\rangle$:

$$\Gamma_{n \to n-1} = \frac{x_{ZPF}^2}{\hbar^2} \, n \, S_{FF}(\omega_0)$$



$\propto S_{FF}(\omega_0)$

Fermi's Golden Rule

See Rev. Mod. Phys. 82, 1155 (2010) for details.

What is S_{FF} from radiation pressure in the optical cavity? Classically, we wrote:

photons per time $\frac{N}{\Delta T}$

$$\mathcal{F} = \frac{2 n h \nu}{c} = \frac{2 N h \nu}{c \Delta t}$$

angular freq.

$$\mathcal{F} = \frac{\hbar \omega_c}{c \frac{\Delta t}{2}} N = \frac{\hbar \omega_c}{L} N$$

cavity length

Quantum :

$$\hat{F} = \left(\frac{\hbar \omega_c}{L} \right) \hat{n}_c$$

number operator for cavity photons

For a fluctuating force $\hat{F}$ acting on a quantum harmonic oscillator, we can say that we have extra term in the Hamiltonian :

$$\hat{U} = - \hat{x} \hat{F} = - x_{ZPF} \left(\hat{a}^{+} + \hat{a} \right) \hat{F}$$

This fluctuating force will cause transitions between oscillator eigenstates. The corresponding Fermi Golden rule rates come from an application of time-dependent perturbation theory and are related to $S_{FF}(\omega)$.

As shown before the transition rate from oscillator eigenstate $|n\rangle$ to $|n+1\rangle$ is :

$$\Gamma_{n \to n+1} = \frac{x_{ZPF}^2}{\hbar^2}(n+1) S_{FF}(-\omega_0) = (n+1)\Gamma_\uparrow$$

and from $|n\rangle$ to $|n-1\rangle$ is :

$$\Gamma_{n \to n-1} = \frac{x_{ZPF}^2}{\hbar^2} n \, S_{FF}(\omega_0) = n \, \Gamma_\downarrow$$

Given these rates, we can write a master equation for the probability of being in oscillator state $|n\rangle$: $p_n(t)$.

transitions into $|n\rangle$

$$\frac{dp_n}{dt} = \left[n \Gamma_\uparrow p_{n-1} + (n+1)\Gamma_\downarrow p_{n+1} \right] - \left[n \Gamma_\downarrow p_n + (n+1)\Gamma_\uparrow p_n \right]$$

transition out of $|n\rangle$

Note that the average energy of the oscillator is :

$$\langle E(t) \rangle = \sum_{n=0}^{\infty} \hbar\omega_0 \left(n + \tfrac{1}{2}\right) p_n(t)$$

Its time derivative is then :

$$\frac{d\langle E \rangle}{dt} = \sum_{n=0}^{\infty} \hbar\omega_0 \left(n + \tfrac{1}{2}\right) \frac{dp_n}{dt}$$

$$\frac{d\langle E\rangle}{dt} = \sum_{n=0}^{\infty} -\hbar\omega_0\left(n+\tfrac{1}{2}\right)\left(n\Gamma_{\downarrow}p_n + (n+1)\Gamma_{\uparrow}p_n\right)$$

$$+\sum_{m=0}^{\infty} \hbar\omega_0\left(m+\tfrac{1}{2}+1\right)(m+1)\Gamma_{\uparrow}p_m$$

We replaced: $m = n-1$

$n = m+1$

$$+\sum_{q=0}^{\infty} \hbar\omega_0\left(q+\tfrac{1}{2}-1\right) q\,\Gamma_{\downarrow}p_q$$

We replaced: $q = n+1$

$n = q-1$

$$\frac{d\langle E\rangle}{dt} = \sum_{n=0}^{\infty} \hbar\omega_0(n+1)p_n\Gamma_{\uparrow} - \sum_{n=0}^{\infty}\hbar\omega_0\, n\, p_n\Gamma_{\downarrow}$$

$$\frac{d\langle E\rangle}{dt} = \underbrace{\sum_{n=0}^{\infty}\hbar\omega_0\left(n+\tfrac{1}{2}\right)p_n}_{\langle E\rangle}\left[\Gamma_{\uparrow}-\Gamma_{\downarrow}\right]$$

$$+\underbrace{\sum_{n=0}^{\infty}\hbar\omega_0\, p_n\,\tfrac{1}{2}}_{\frac{\hbar\omega_0}{2}}\left[\Gamma_{\uparrow}+\Gamma_{\downarrow}\right]$$

$$\frac{d\langle E\rangle}{dt} = \frac{\hbar\omega_0}{2}\left[\Gamma_{\downarrow}+\Gamma_{\uparrow}\right] - \langle E\rangle\left[\Gamma_{\downarrow}-\Gamma_{\uparrow}\right]$$

$$\frac{d\langle E\rangle}{dt} = \frac{\hbar\omega_0}{2}\frac{x_{ZPF}^2}{\hbar^2}\Big(S_{FF}(\omega_0) + S_{FF}(-\omega_0)\Big)$$

$$- \langle E\rangle \frac{x_{ZPF}^2}{\hbar^2}\Big(S_{FF}(\omega_0) - S_{FF}(-\omega_0)\Big)$$

$$\frac{d\langle E\rangle}{dt} = \overbrace{\frac{1}{4m}\Big[S_{FF}(\omega_0) + S_{FF}(-\omega_0)\Big]}^{P}$$

$$- \frac{\langle E\rangle}{m}\underbrace{\frac{1}{2\hbar\omega_0}\Big[S_{FF}(\omega_0) - S_{FF}(-\omega_0)\Big]}_{\Gamma}$$

heating

dissipation

$$\frac{d\langle E\rangle}{dt} = P - \left(\frac{\Gamma}{m}\right)\langle E\rangle$$

P represents the heating of the oscillator by the noise source and $\left(\frac{\Gamma}{m}\right)$ represents the damping of the oscillator by the noise source. The heating effect is the result of a random force causing the oscillator's momentum to diffuse, causing $\langle E \rangle$ to grow linearly in time. This is due to the symmetric in frequency part of the quantum spectral density $\left(S_{FF}(\omega_0) + S_{FF}(-\omega_0)\right)$. The damping effect is caused by the net tendency of the noise to absorb energy from, rather than emit energy to the oscillator. This is due to the asymmetric in frequency part of the quantum spectral density $\left(S_{FF}(\omega_0) - S_{FF}(-\omega_0)\right)$.

This yields the quantum version of the fluctuation dissipation theorem. If the system is in thermal equilibrium, then the transition rates must satisfy the detailed balance relation :

$$\Gamma_{n \to n+1} = \Gamma_{n \to n-1}$$

$$(\bar{n}+1)\, S_{FF}(-\omega_0) = \bar{n}\, S_{FF}(\omega_0)$$

$$\frac{\bar{n}+1}{\bar{n}} = \frac{S_{FF}(\omega_0)}{S_{FF}(-\omega_0)}$$

$$1 + \frac{1}{\bar{n}} = \frac{S_{FF}(\omega_0)}{S_{FF}(-\omega_0)}$$

Recall that at thermal equilibrium,

$$\bar{E} = \hbar\omega_0 \left(\frac{1}{2} + \underbrace{\frac{1}{e^{\frac{\hbar\omega_0}{k_B T}} - 1}}_{\bar{n}} \right) = \hbar\omega_0 \left(\bar{n} + \frac{1}{2} \right)$$

Therefore:

$$e^{\frac{\hbar\omega_0}{k_B T}} = \frac{S_{FF}(\omega_0)}{S_{FF}(-\omega_0)}$$

This allows us to relate the symmetric and asymmetric spectral densities:

$$\frac{S_{FF}(\omega_0) + S_{FF}(-\omega_0)}{S_{FF}(\omega_0) - S_{FF}(-\omega_0)} = \frac{1 + e^{-\frac{\hbar\omega_0}{k_B T}}}{1 - e^{-\frac{\hbar\omega_0}{k_B T}}}$$

$$= \coth\left(\frac{\hbar\omega_0}{2 k_B T} \right)$$

$$\underbrace{\left[S_{FF}(\omega_0) + S_{FF}(-\omega_0)\right]}_{\bar{S}_{FF}(\omega_0)} = \coth\left(\frac{\hbar\omega_0}{2k_BT}\right) \underbrace{\left[S_{FF}(\omega_0) - S_{FF}(-\omega_0)\right]}_{2\hbar\omega_0\Gamma(\omega_0)}$$

$\bar{S}_{FF}(\omega_0)$ ← quantum version of single-sided PSD $\bar{S}_F(\omega_0)$.

symmetric quantum spectral density

$2\hbar\omega_0\Gamma(\omega_0)$

quantum definition of dissipation

$$\boxed{\bar{S}_{FF}(\omega_0) = \coth\left(\frac{\hbar\omega_0}{2k_BT}\right) 2\hbar\omega_0\Gamma(\omega_0)}$$

Noise and dissipation are related by temperature in equilibrium.
This is the quantum version of the Fluctuation - Dissipation theorem.

For high temperatures, $k_BT \gg \hbar\omega_0$ we recover the classical fluctuation-dissipation theorem:

$$\bar{S}_{FF}(\omega_0) = \frac{1 + e^{-\frac{\hbar\omega_0}{k_BT}}}{1 - e^{-\frac{\hbar\omega_0}{k_BT}}} 2\hbar\omega_0\Gamma(\omega_0)$$

$$\approx \frac{2}{\frac{\cancel{\hbar\omega_0}}{k_BT}} 2\cancel{\hbar\omega_0}\Gamma(\omega_0)$$

$$\boxed{\bar{S}_{FF}(\omega_0) \approx 4k_BT\Gamma(\omega_0)}$$

✓

Note that the quantum version of the Fluctuation - Dissipation Theorem can be rewritten in terms of the average occupation number $\bar{n}$, since:

$$\bar{n} = \frac{1}{e^{\frac{\hbar\omega_0}{k_B T}} - 1}$$

$$2\bar{n} + 1 = \frac{2 + e^{\frac{\hbar\omega_0}{k_B T}} - 1}{e^{\frac{\hbar\omega_0}{k_B T}} - 1} = \frac{e^{\frac{\hbar\omega_0}{k_B T}} + 1}{e^{\frac{\hbar\omega_0}{k_B T}} - 1}$$

$$2\bar{n} + 1 = \coth\left(\frac{\hbar\omega_0}{2k_B T}\right)$$

$$\therefore \boxed{\bar{S}_{FF}(\omega_0) = 2\hbar\omega_0 (2\bar{n} + 1)\, \Gamma(\omega_0)}$$

This expression also allows us to solve for the average occupation number given a known quantum spectral density of force fluctuations.

$$2\bar{n} + 1 = \frac{1}{2\hbar\omega_0} \frac{\bar{S}_{FF}(\omega_0)}{\Gamma(\omega_0)}$$

$$2\bar{n} + 1 = \frac{S_{FF}(\omega_0) + S_{FF}(-\omega_0)}{S_{FF}(\omega_0) - S_{FF}(-\omega_0)}$$

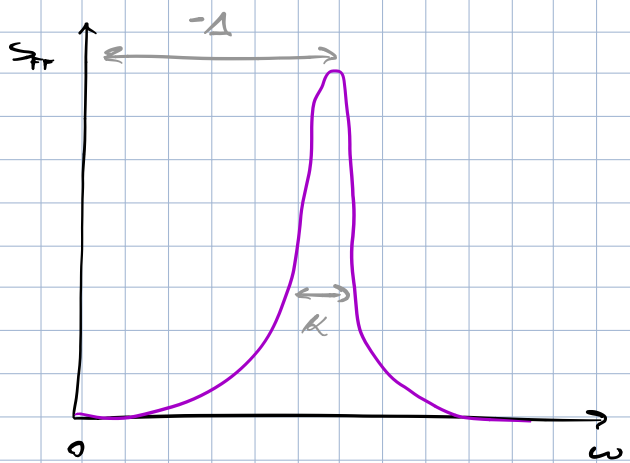
$$\bar{n} = \frac{S_{FF}(-\omega_0)}{S_{FF}(\omega_0) - S_{FF}(-\omega_0)}$$

(1) $$\frac{1}{\bar{n}} = \frac{S_{FF}(\omega_0)}{S_{FF}(-\omega_0)} - 1$$

Given a quantum force noise spectral density, we can now see what its effect will be on the mechanical mode's average occupation number $\bar{n}$.

The noise spectral density of this optical force is given by:

(2) $$S_{FF}(\omega) = \left(\frac{\hbar\omega_c}{L}\right)^2 \bar{n}_c \underbrace{\frac{\kappa}{(\omega+\Delta)^2 + \left(\frac{\kappa}{2}\right)^2}}_{\text{photon shot noise spectrum}}$$



$\Delta = \omega_L - \omega_c$

- κ is the cavity decay rate
- $\Delta = \omega_L - \omega_c$ is the cavity detuning

Assuming that the force noise due to the cavity dominates over other sources (this implies that optical damping dominates over intrinsic damping $\Gamma_{opt} >> \Gamma$), we can use (1) and (2) to solve for the equilibrium occupation number $\bar{n}$:

$$\frac{1}{\bar{n}} = \frac{(-\omega_0 + \Delta)^2 - (\omega_0 + \Delta)^2}{(\omega_0 + \Delta)^2 + \left(\frac{\kappa}{2}\right)^2}$$

$$\bar{n} = -\frac{(\omega_0 + \Delta)^2 + \left(\frac{\kappa}{2}\right)^2}{4\omega_0 \Delta}$$

The phonon occuption $\bar{n}$ is minimum for a cavity detuning given by solving $\frac{d\bar{n}}{d\Delta} = 0$ for Δ:

$$\Delta_{min} = -\sqrt{\omega_0^2 + \left(\frac{\kappa}{2}\right)^2}$$

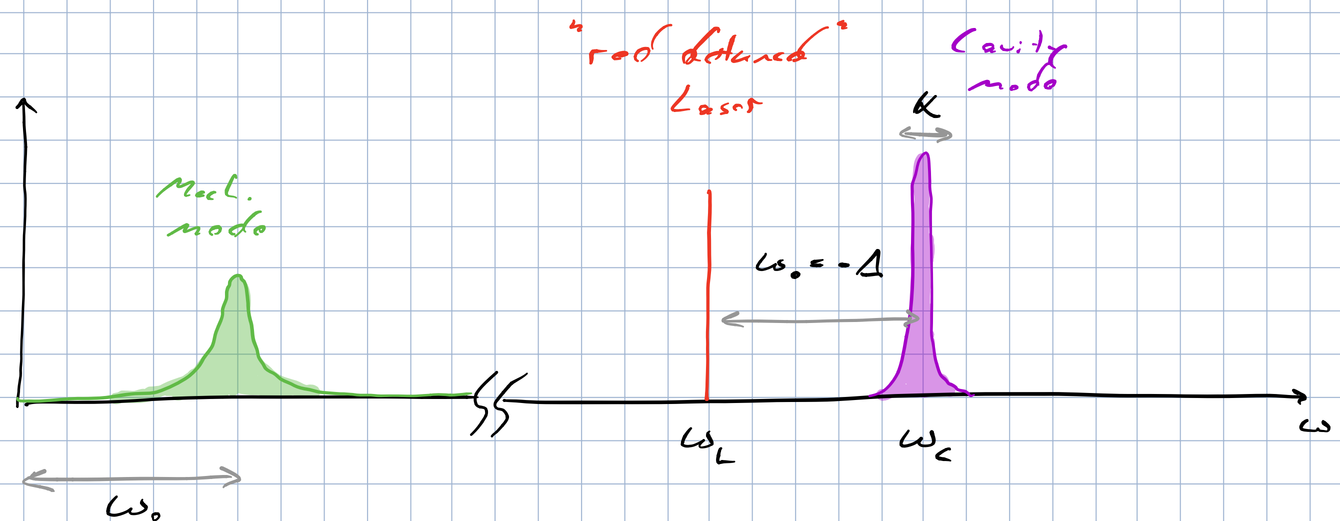
For high-finesse cavity, we have the so-called "resolved sideband" regime : $\omega_0 >> \kappa$:

$$\boxed{\begin{aligned}&\Delta_{min} \approx -\omega_0\\ &\hookrightarrow \quad \bar{n}_{min} = \left(\frac{\kappa}{4\omega_0}\right)^2\end{aligned}}$$

$\hookrightarrow$ therefore to cool the mechanical resonator, the linewidth κ should be small and the mechanical frequency ω_0 should be large.

Graphical Explanation

Cooling mechanical mode :

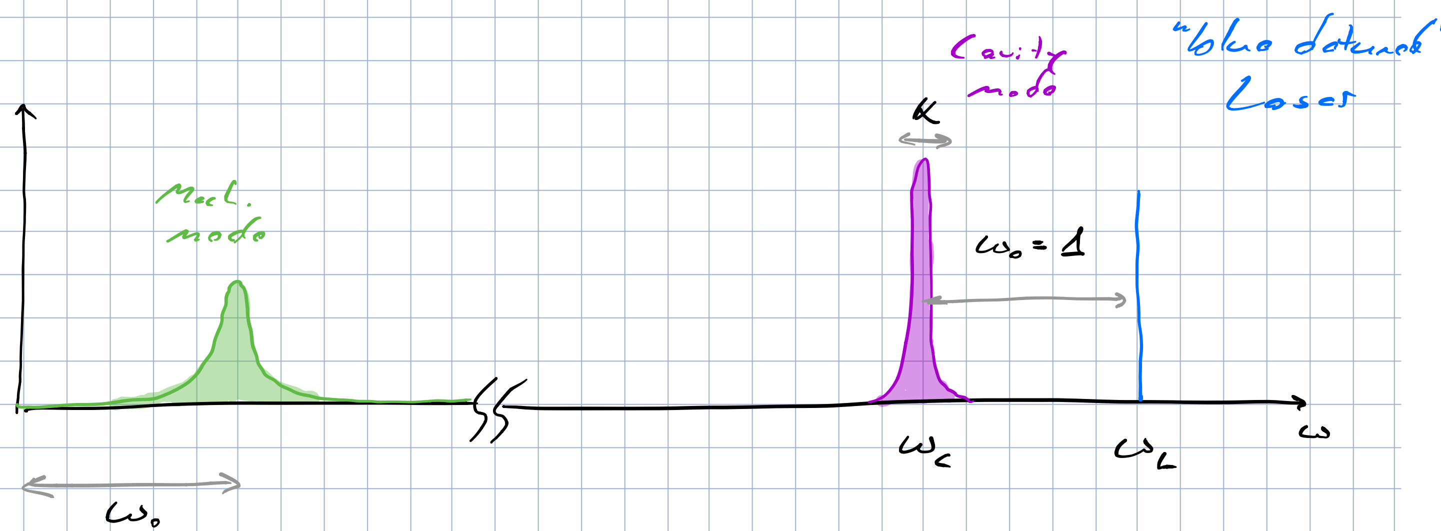


$\Delta = \omega_L - \omega_c$

A laser photon (ω_L) combines with a phonon from the resonator (ω_0) to produce a cavity photon (ω_c). This extracts phonons from resonator and releases photons in the cavity.

The reverse can also be done for a detuning of $\Delta = \omega_0$. Here a laser photon (ω_L) excites a phonon from the resonator (ω_0) and a cavity photon (ω_c).

Heating mechanical mode:



$$\Delta = \omega_L - \omega_c$$