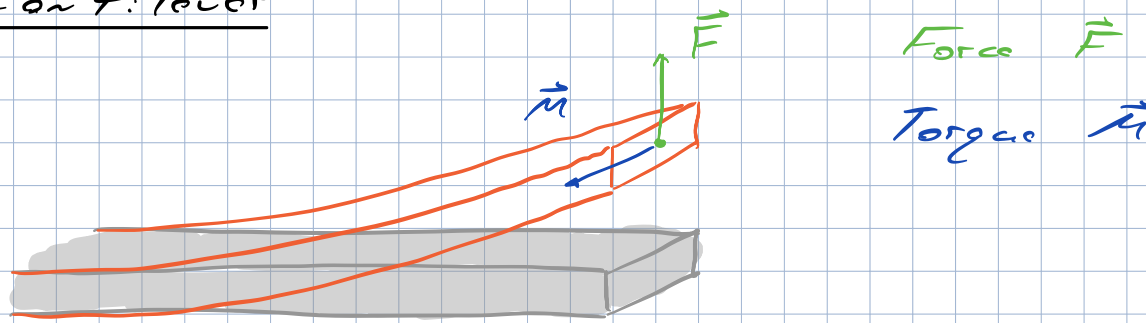
Lecture 1 (16.09.2020)

Cantilever Basics

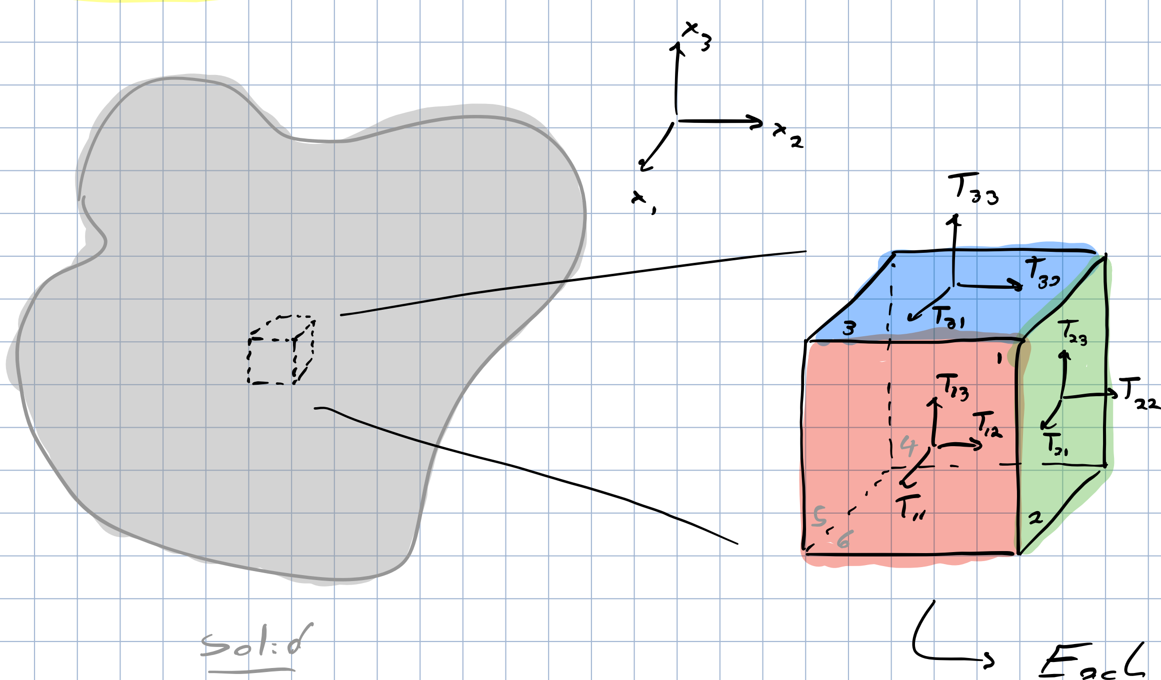
I. Cantilever



- Useful and representative mechanical transducer. Others include doubly-clamped beams, strings, membranes, etc.
- Need to introduce deformation of solids in order to discuss their motion.

II. Stress and Strain (FN p. 145)

Stress:



↳ Each face has area A.

Force on each surface : $\vec{F}_i = \sum_{j=1}^{3} F_{ij} \hat{x}_j$

where $i = 1$ to 6.

We define a vector stress :

$\vec{f}_i = \underbrace{\frac{\vec{F}_i}{A}}_{\frac{\text{Force}}{\text{area}} = \text{pressure } [N/m^2]}$: $\boxed{\vec{f}_i = \sum_{j=1}^{3} \frac{F_{ij}}{A} \hat{x}_j = \sum_{j=1}^{3} T_{ij} \hat{x}_j}$

stress tensor

We have an infinitesimal cube in <u>static</u> equilibrium. Therefore forces and torques must be uniform and add to zero.

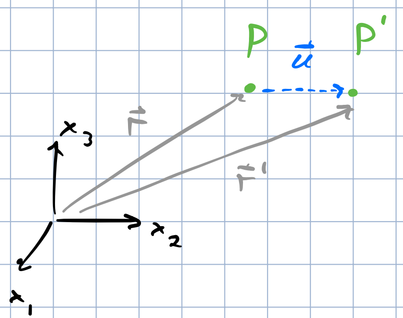
Force balance:

$$\left.\begin{aligned} \vec{F}_1 &= -\vec{F}_4 \\ \vec{F}_2 &= -\vec{F}_5 \\ \vec{F}_3 &= -\vec{F}_6 \end{aligned}\right\} \therefore \text{ we can just consider } i = 1 \text{ to } 3$$

Torque balance: $\vec{M}_{tot} = 0 \;\} \;\therefore\; T_{ij} = T_{ji}$

Strain :

Such a stress applied to a solid can result in deformation, i.e. strain.



The local deformation of a solid is quantified by the relative displacement vector $\vec{u}$ of a point in that solid.

Spatial derivatives of this displacement define the strain tensor :

$$\boxed{S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}$$

General Stress - Strain Relations : (EN p. 191)

$$T_{ij} = \sum_{kl} \alpha_{ijk} S_{kl}$$

The constants α_{ijk} depend on material parameters :

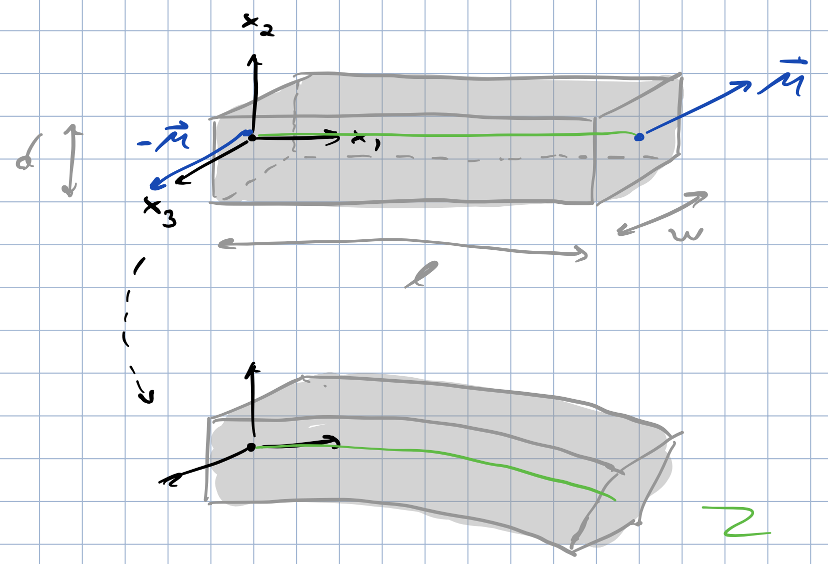
stiffness

E = Young's modulus

ν = Poisson's Ratio

ratio of contraction $\perp$ to applied load

III. Example: Bending by Pure Torque (FN p. 156)



$\leftarrow$ neutral axis $(x_2 = x_3 = 0)$

Static case:

$$\vec{M}(x_1 = \ell) = -M_0\,\hat{x}_3$$

$$\vec{M}(x_1 = 0) = M_0\,\hat{x}_3$$

One way to apply this type of torque is with the surface stress:

$$\vec{f}(0, x_2, x_3) = -t_0\,x_2\,\hat{x}_1$$

$$\vec{f}(\ell, x_2, x_3) = t_0\,x_2\,\hat{x}_1$$

In this case, we need to have:

$$\vec{M}(x_1 = \ell) = \int_{-\frac{d}{2}}^{\frac{d}{2}}\int_{-\frac{w}{2}}^{\frac{w}{2}} \underbrace{\vec{r}\times\vec{f}(\ell, x_2, x_3)}_{t_0\left(-x_2^2\,\hat{x}_3 + \cancel{x_2 x_3\,\hat{x}_2}^{\,0}\right)}\,dx_3\,dx_2$$

$$\vec{M}(x_1 = \ell) = \int_{-\frac{d}{2}}^{\frac{d}{2}}\int_{-\frac{w}{2}}^{\frac{w}{2}} \left(-t_0\,x_2^2\,\hat{x}_3\right) dx_3\,dx_2 = -\frac{1}{12}\,w\,d^3\,t_0\,\hat{x}_3$$

$$\therefore\quad M_0 = \frac{w\,d^3}{12}\,t_0$$

$$t_0 = \frac{12 M_0}{w d^3}$$

or

$$t_0 = \frac{M_0}{I_3} \quad \text{with} \quad I_3 = \frac{w d^3}{12}$$

where 2^{nd} moments of inertia are:

$$I_3 = \int x_2^2 \, dA$$

$$I_2 = \int x_3^2 \, dA$$

In terms of our stress tensor, we have:

$$T_{11} = t_0 x_2 = \frac{M_0}{I_3} x_2$$

All other terms are zero.

By applying the stress-strain relations and boundary conditions, we can find the resulting deformation (FN p. 199):

$$u_1 = \frac{M_0}{I_3} \frac{\nu}{E} x_1 x_2$$

$$u_2 = -\frac{M_0}{I_3} \frac{1}{2E} \left(x_1^2 + \nu x_2^2 - \nu x_3^2 \right)$$

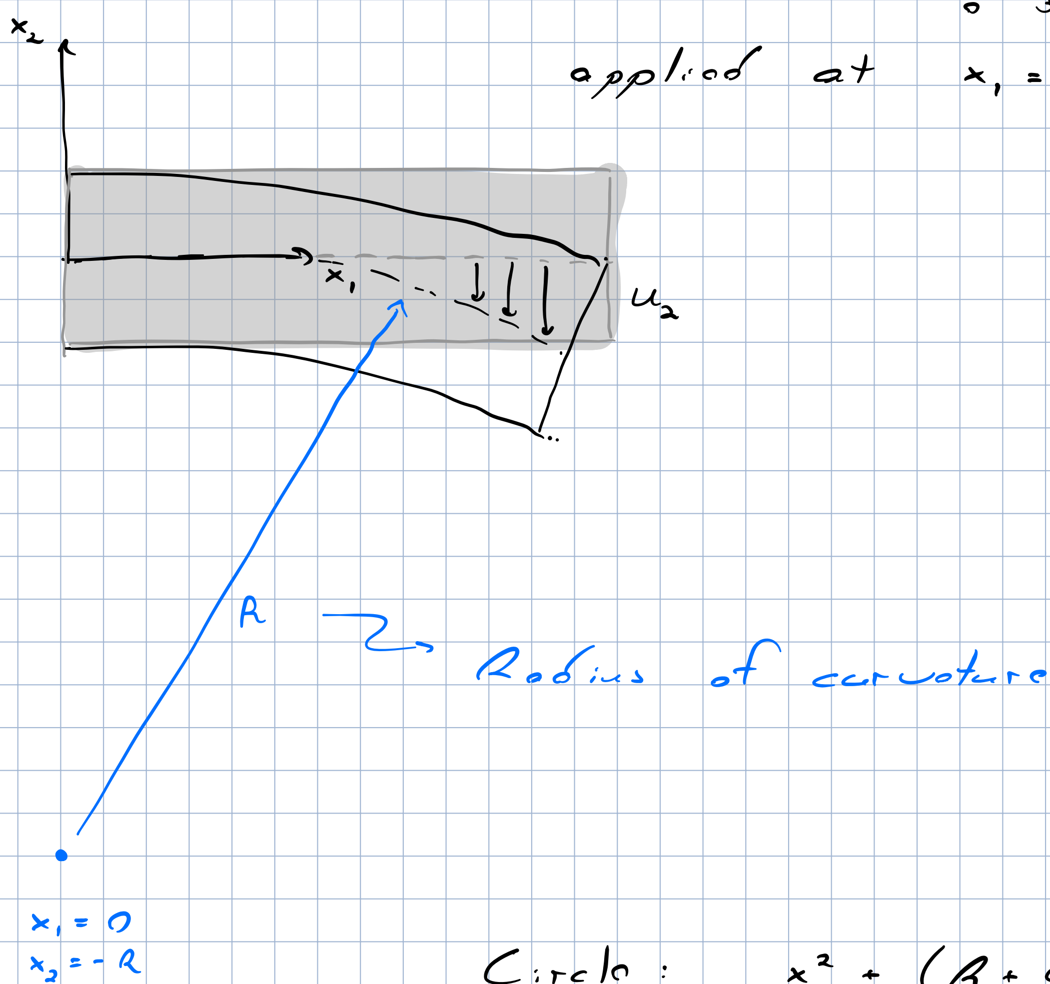
$$u_3 = -\frac{M_0}{I_3} \frac{\nu}{E} x_2 x_3$$

Along the neutral axis $(x_2 = x_3 = 0)$:

$$\boxed{u_2 = -\frac{M_0}{2EI_3} x_1^2} \quad (1)$$

For $\vec{M} = -M_0 \hat{x}_3$

applied at $x_1 = \ell$



Circle : $x_1^2 + (R + u_2)^2 = R^2$

$$u_2 = \sqrt{R^2 - x_1^2} - R$$

$$u_2 = R\sqrt{1 - \left(\frac{x_1}{R}\right)^2} - R$$

For $x_1 << R$:

$$u_2 \approx R\left(1 - \frac{1}{2}\left(\frac{x_1}{R}\right)^2\right) - R$$

$$\boxed{u_2 \approx -\frac{x_1^2}{2R}} \quad (2)$$

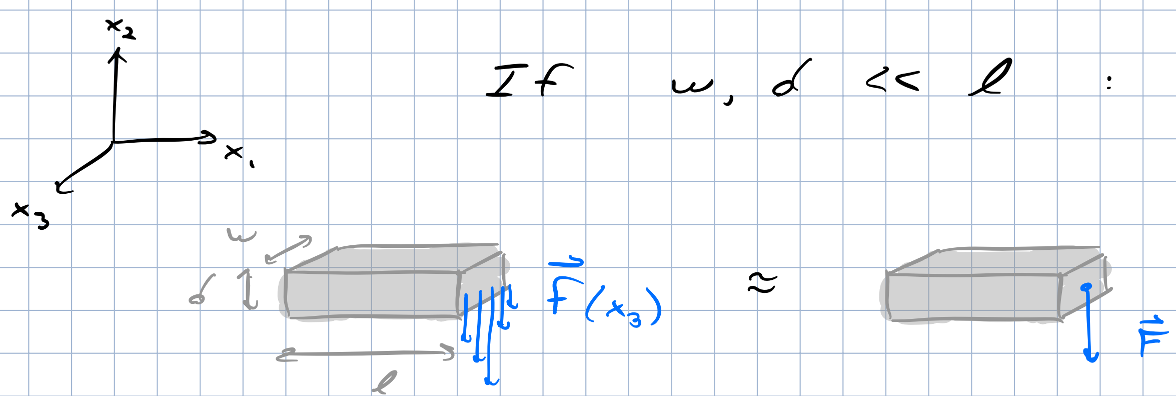
Putting (1) and (2) together:

$$\boxed{R \approx \frac{EI_3}{M_0}}$$

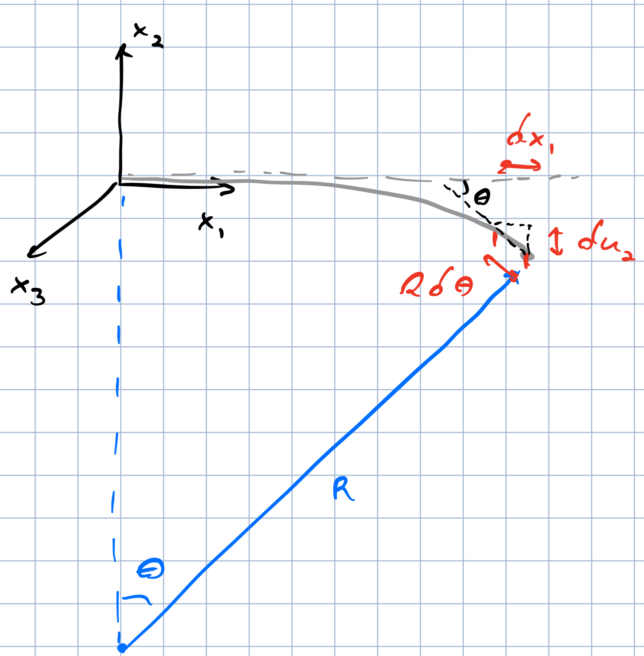
IV. Euler - Bernoulli Theory of Beams

- Saint - Venant's Principle:

If $w, d \ll \ell$:



- Local Radius of Curvature:



$R \gg \ell$

$R \gg R\theta$

$\theta \ll 1$

$$\frac{du_2}{dx_1} = -\tan\theta \approx -\theta$$

$$\frac{d^2u_2}{dx_1^2} \approx -\frac{d\theta}{dx_1} \qquad dx_1 \approx R\,d\theta$$

$$\frac{d^2 u_2}{d x_1^2} \approx -\frac{1}{R} = -\frac{M_0}{E I_3}$$

In Euler-Bernoulli:

Limit :

$$\boxed{\frac{d^2 u_2}{d x_1^2} \approx -\frac{M_0}{E I_3}}$$