Introduction to Nanomechanics Spring Semester 2012
Exercise Sheet 3 submission due on Monday, 26.03.2012, 1:00 pm

1 The Quantum Harmonic Oscillator

The motion of a cold nanomechanical resonator is assumed to be treated through the standard quantum harmonic oscillator Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hbar\omega\left(\frac{1}{2} + \hat{a}^\dagger\hat{a}\right) = \hbar\omega\left(\frac{1}{2} + \hat{N}\right) \tag{1}$$

where $\hat{H}$, $\hat{p}$, $\hat{x}$, $\hat{a}$, $\hat{a}^\dagger$, and $\hat{N}$ are, respectively, the hamiltonian, momentum, position, annihilation, creation and number operators.

1. Derive the commutation relation for $\hat{a}$ and $\hat{a}^\dagger$ using the commutation relation for the canonically conjugate operators $\hat{x}$, $\hat{p}$ and the following definitions: (1 point)

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p}) ; \quad \hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p}) . \tag{2}$$

2. Reminding that the effects of the operators $\hat{a}$ and $\hat{a}^\dagger$ on a state $|n\rangle$ are the following:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle , \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle , \tag{3}$$

evaluate the expectation values $\langle n|\hat{x}|n\rangle$ and $\langle n|\hat{x}^2|n\rangle$. (1 point)

3. The root mean square amplitude of quantum fluctuations of the resonator position in its ground state is also called the *Standard Quantum Limit*:

$$\Delta x_{SQL} \equiv \sqrt{\langle 0|\hat{x}^2|0\rangle}. \tag{4}$$

Calculate this value for the harmonic oscillator. (0.5 points)

4. In quantum mechanics, the virial theorem establishes a relation between the eigenvalues of the kinetic energy operator and those of the potential energy operator, both calculated with respect to the eigenvectors of the Hamiltonian of the system. Write down this expression in the case of the harmonic oscillator and use it to calculate the root mean square amplitude of quantum fluctuations of the oscillator momentum, in its ground state. (1 point)

5. Linking together the results of the last two questions, verify that the ground state of the harmonic oscillator is a state of minimum indetermination. (1 point)

6. Consider now and for the following questions the case of a harmonic oscillator in equilibrium with its environment at a finite temperature T. Apply the equipartition theorem to determine the mean total energy $\bar{E}$. Compare this result with the case of a free particle moving in a 3-dimensional space. (2 points)

7. Calculate the *rms* displacement fluctuations of the resonator $\sqrt{\langle x_{th}^2 \rangle}$. (0.5 points)

8. Calculate the probability $P_{th}(n)$ for the resonator to be found in a generic state $|n\rangle$. (2 points)

9. In the low-temperature limit $k_B T \ll \hbar\omega$, write down the expression of the average state occupation number $\bar{N}$ (or N_{th}) (*you do not need to demonstrate it*). Derive the relation for the probability $P_{th}(0)$ of the resonator to be found in the ground state in terms of $\bar{N}$ and give a numerical estimate. (1 point)