Let's start by discussing quantum noise.
Similar to the classical spectral density, we define a quantum spectral density:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \langle \hat{x}(t)\, \hat{x}(0) \rangle$$

$\hat{x}$ is a quantum operator
$\langle\ \rangle$ is a quantum statistical average using a density matrix

$\rightarrow \frac{d\hat{A}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}] + \frac{\partial A}{\partial t}$

Solutions to the Heisenberg equations of motion for a harmonic osc give:

$$\hat{x}(t) = \hat{x}(0)\cos(\omega_0 t) + \hat{p}(0)\frac{1}{m\omega_0}\sin(\omega_0 t)$$
$$\hat{p}(t) = \hat{p}(0)\cos(\omega_0 t) - \hat{x}(0)\, m\omega_0 \sin(\omega_0 t)$$

$$\therefore \quad \langle \hat{x}(t)\,\hat{x}(0)\rangle = \langle \hat{x}(0)\,\hat{x}(0)\rangle \cos(\omega_0 t) + \langle \hat{p}(0)\,\hat{x}(0)\rangle \frac{1}{m\omega_0}\sin(\omega_0 t)$$

Recall, however, that: $[\hat{x}(0), \hat{p}(0)] = i\hbar$

$$\hat{x} = x_{ZPF}(\hat{a}^+ + \hat{a})$$
$$\hat{p} = \frac{i\hbar}{2x_{ZPF}}(\hat{a}^+ - \hat{a})$$

w/ $[\hat{a}, \hat{a}^+] = 1$

$\hat{n} = \hat{a}^+\hat{a}$

w/ $x_{ZPF}^2 = \langle 0|\,\hat{x}^2\,|0\rangle = \frac{\hbar}{2m\omega_0}$

For thermal equilibrium:

$$\langle \hat{x}(0)\,\hat{p}(0)\rangle = +i\frac{\hbar}{2}$$
$$\langle \hat{p}(0)\,\hat{x}(0)\rangle = -i\frac{\hbar}{2}$$

So, the correlator is:

$$\langle \hat{x}(t)\,\hat{x}(0)\rangle = \underbrace{\langle \hat{x}(0)\,\hat{x}(0)\rangle}\,\frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2} + \underbrace{\langle \hat{p}(0)\,\hat{x}(0)\rangle}\,\frac{1}{m\omega_0}\,\frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i}$$

$$\rightarrow \hat{x}\,\hat{x} = x_{zpf}^2\left(\cancel{\hat{a}^\dagger\hat{a}^\dagger} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right)$$

$$\rightarrow \hat{p}\,\hat{x} = \frac{i\hbar}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger - \hat{a}\hat{a}\right)$$

$$\langle \hat{x}(t)\,\hat{x}(0)\rangle = x_{zpf}^2\left[\langle(2\hat{n}+1)\rangle\,\frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2} - \langle(1)\rangle\,\frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2}\right]$$

$$\langle \hat{x}(t)\,\hat{x}(0)\rangle = x_{zpf}^2\left[\langle\hat{n}\rangle\,e^{i\omega_0 t} + \langle\hat{n}+1\rangle\,e^{-i\omega_0 t}\right] \Longleftarrow$$

$\therefore$ The spectral density is therefore:

(1) $$S_{xx}(\omega) = 2\pi\,x_{zpf}^2\left\{\langle\hat{n}\rangle\,\delta(\omega+\omega_0) + \langle\hat{n}+1\rangle\,\delta(\omega-\omega_0)\right\}$$

stimulated emission into the osc.

(absorption)

emission by the osc

(emission)

In the high temperature limit:

$$k_B T \gg \hbar\Omega$$

$$\langle\hat{n}\rangle \sim \langle\hat{n}+1\rangle \sim \frac{k_B T}{\hbar\omega_0}$$

$$S_{xx}(\omega) = \pi\,\frac{k_B T}{m\omega_0^2}\left\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right\}$$

Let's take a general cavity (optical or RF) coupled to the position of an oscillator.

Changes in the position of the oscillator lead to changes in the phase of the cavity's reflected carrier signal. The changing phase shift can be converted to a changing intensity (by interferometry).

Applies to microwave cavities, optical cavities, etc.

cavity → • uses wave nature of light for the interference signal

• suffers from particle nature of light in the noise.

A coherent photon state contains a Poisson distribution of the number of photons:

$$\underset{\text{flucts}}{(\Delta N)^2} = \underset{\text{mean}}{\bar{N}}$$

$$\underset{\text{phase uncertainty}}{(\Delta\Theta)^2} = \frac{1}{4\bar{N}}$$

$\Longrightarrow$ $\Delta N \Delta\Theta = \frac{1}{2}$ Number-phase uncertainty relation.

In spectral densities this gives:

$$S_{nn} S_{\theta\theta} = \frac{1}{4}$$

spectral density of phase fluct.

spectral density of photon flux

$$\sqrt{S_{nn} S_{\theta\theta}} = \frac{1}{2}$$

Take the case of light reflecting off a mirror. The beam will have a phase shift $2kx$ if the mirror moves x. The phase uncertainty leads to a position imprecision:

$$S_{xx}^{I} = \frac{1}{4k^2} S_{\theta\theta}$$

The back action imparted by a photon hitting the mirror is $2\hbar k$. So the photon shot noise corresponds to a random back action force:

$$S_{FF} = 4\hbar^2 k^2 S_{nn}$$

Together:

$$S_{FF} S_{xx}^{I} = \hbar^2 S_{nn} S_{\theta\theta} = \frac{\hbar^2}{4} \qquad (2)$$

$$\underbrace{\sqrt{S_{FF} S_{xx}^{I}} = \frac{\hbar}{2}}_{\text{Quantum limit on noise of detector}}$$

Let's apply all of this to our mechanical oscillator in a resonant cavity.

→ Standard Quantum Limit for Position Measurement

(weak measurement — many cycles of oscillation before information is acquired)

$$\hat{x}(t) = \hat{X}\cos(\omega_0 t) + \hat{Y}\sin(\omega_0 t)$$

$$[\hat{X}, \hat{Y}] = \left[\hat{x}(0), \frac{1}{m\omega_0}\hat{p}(0)\right] = \frac{1}{m\omega_0}\left[\hat{x}(0), \hat{p}(0)\right] = \frac{i\hbar}{m\omega_0}$$

$$[\hat{X}, \hat{Y}] = 2i\,x_{ZPF}^2$$

B/c this is a weak measurement, we are trying to measure two incompatible observables simultaneously.

Spectral Density of harmonic oscillator w/ resonant frequency ω_0 & damping γ_0 & equation of motion:

$$m\ddot{x} + m\gamma_0\dot{x} + m\omega_0^2 x = F$$

$$x(t) = \frac{1}{\sqrt{2\pi}}\int x(\omega)\, e^{-i\omega t}\, d\omega$$

is

$$S_{xx}(\omega) = x_{ZPF}^2\left\{\langle n\rangle \frac{\gamma_0}{(\omega+\omega_0)^2 + \left(\frac{\gamma_0}{2}\right)^2} + \langle n+1\rangle \frac{\gamma_0}{(\omega-\omega_0)^2 + \left(\frac{\gamma_0}{2}\right)^2}\right\}$$

where we have referred back to (1) on page 2 and converted δ-functions into Lorentzians.

For the classical limit ($k_B T \gg \hbar\omega_0$):

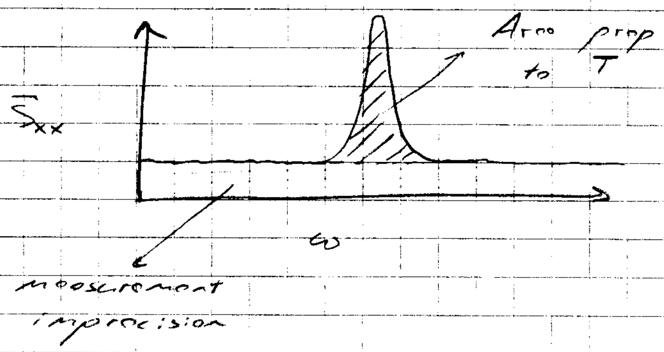
$$\bar{S}_{xx}(\omega) = \frac{1}{2}\left[S_{xx}(\omega) + S_{xx}(-\omega)\right]$$

$$\approx \frac{k_B T}{2m\omega_0^2}\,\frac{\gamma_0}{(|\omega|-\omega_0)^2 + \left(\frac{\gamma_0}{2}\right)^2}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{S}_{xx}(\omega) = \frac{k_B T}{m\omega_0^2}$$

$$\therefore \quad m\omega_0^2 \langle x^2 \rangle = k_B T$$

$$\hookrightarrow \quad \frac{1}{2} m\omega_0 \langle x^2 \rangle = \frac{1}{2} k_B T$$ ← Equipartition



At zero temperature, we have:

$$\bar{S}_{xx}^{0} = x_{ZPF}^2 \frac{\gamma_0/2}{(|\omega| - \omega_0)^2 + (\gamma_0/2)^2}$$

One might expect to measure this, but this neglects the effect of back-action.
A displacement $\delta x(\omega)$ will be caused by a force $F(\omega)$ through the mechanical susceptibility $\chi_{xx}(\omega)$.

From: $m\ddot{x} + m\gamma_0\dot{x} + m\omega_0^2 x = F$

$$\delta x(\omega) = \chi_{xx}(\omega) F(\omega)$$

$$\text{w/} \quad \chi_{xx}(\omega) = \frac{1}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma_0\omega}$$

So, extra displacement will be caused by the back-action of the detector.

Let's look at the resonant freq. of the oscillator, $\omega = \omega_0$:

$$\bar{S}_{xx,tot}(\omega_0) = \bar{S}^{0}_{xx}(\omega_0) + \frac{1}{2}|\chi_{xx}(\omega_0)|^2 \left(S_{FF}(\omega_0) + S_{FF}(-\omega_0)\right) + \frac{1}{2}\left(S^{I}_{xx}(\omega_0) + S^{I}_{xx}(-\omega_0)\right)$$

back-action

measurement imprecision

$$\bar{S}_{xx,tot}(\omega_0) = \bar{S}^{0}_{xx}(\omega_0) + \bar{S}_{xx,add}(\omega_0)$$

$\bar{S}^{0}_{xx}$ is the zero-point motion :

$$\bar{S}^{0}_{xx}(\omega_0) = \frac{2 x_{ZPF}^2}{\gamma_0}$$

Let's assume the best possible detector — a quantum limited detector. See (2) from ④:

$$S_{FF} S^{I}_{xx} = \frac{\hbar^2}{4}$$

$$\therefore \bar{S}_{xx,add}(\omega_0) = |\chi_{xx}(\omega_0)|^2 S_{FF} + \frac{\hbar^2}{4}\frac{1}{S_{FF}}$$

least additional noise created when :

$$S_{FF,opt} = \frac{\hbar}{2|\chi_{xx}(\omega_0)|} = \frac{\hbar}{2} m\omega_0\gamma_0$$

for large damping, less susceptible to back-action

At optimal point :

$$\bar{S}_{xx,add}(\omega_0) = \frac{\hbar}{m\omega_0\gamma_0} = \bar{S}^{0}_{xx}(\omega_0)$$

The added position noise is exactly equal to the noise power of the zero-point fluctuations.

This limit has nothing to do w/ temperature; this is purely a quantum effect. If you have a perfect detector ($S_{FF} S_{xx}^{I}$ is minimum), this will be the noise limit.

$$\therefore \quad \bar{S}_{xx,tot}(\omega_0) = 2\,\bar{S}^{0}_{xx}(\omega_0) \quad \Longleftarrow$$

total inferred position noise

1/2 noise from the oscillator

1/2 noise from the detector

If we consider all freqs., not just the resonance $\omega = \omega_0$, then:

$$\bar{S}_{xx,tot}(\omega) = x_{ZPF}^2 \frac{\gamma_0/2}{(|\omega|-\omega_0)^2 + (\gamma_0/2)^2} + \frac{\hbar}{2}\left(\frac{|\chi_{xx}(\omega)|^2}{|\chi_{xx}(\omega_0)|} + |\chi_{xx}(\omega_0)| \right)$$

$$\approx \frac{x_{ZPF}^2}{\gamma_0} \left\{ 1 + 3 \frac{(\gamma_0/2)^2}{(|\omega|-\omega_0)^2 + (\gamma_0/2)^2} \right\}$$

$$\frac{d\hat{A}}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}\right] + \frac{\partial \hat{A}}{\partial t}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2}$$

$$\frac{d}{dt}\hat{x} = \frac{i}{\hbar}\left[\hat{H}, \hat{x}\right] = \frac{\hat{p}}{m}$$

$$\frac{d}{dt}\hat{p} = \frac{i}{\hbar}\left[\hat{H}, \hat{p}\right] = -m\omega_0^2 \hat{x}$$

$$\dot{\hat{p}}(0) = -m\omega_0^2 \hat{x}(0)$$

$$\dot{\hat{x}}(0) = \frac{p(0)}{m}$$

$$\hat{x}(t) = \hat{x}(0)\cos(\omega_0 t) + \frac{\hat{p}(0)}{m\omega_0}\sin(\omega_0 t)$$

$$\hat{p}(t) = \hat{p}(0)\cos(\omega_0 t) - m\omega_0 \hat{x}(0)\sin(\omega_0 t)$$