Nanomechanics

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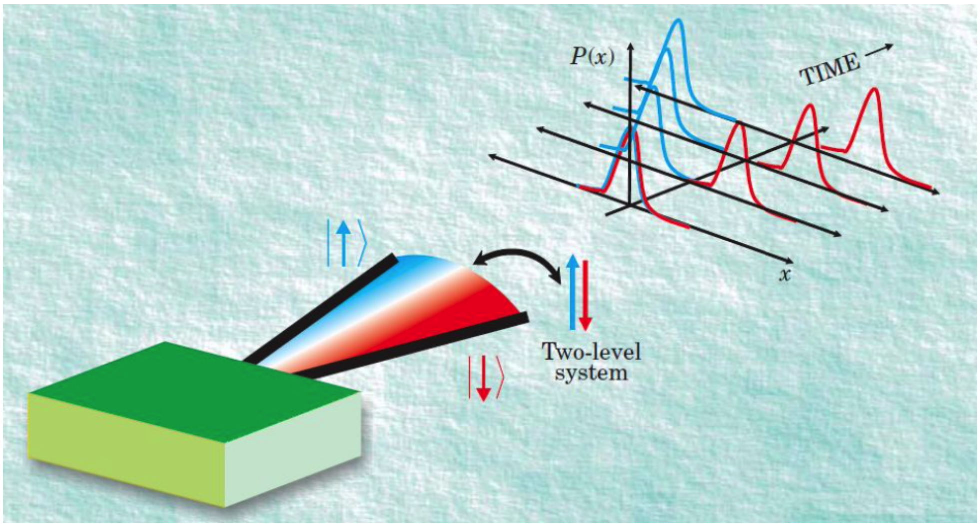


Contents

0 Introduction

This lecture gives an introduction to the physics of nanomechanical devices. These objects show significantly different properties compared to macroscopic ones. For example, small bodies are heavily affected by thermal fluctuations. Considering a spring-like harmonic oscillator with spring constant k in thermal equilibrium at temperature T, the equipartition theorem states that

$$\frac{1}{2}k\langle x^2\rangle = \frac{1}{2}k_B T\,, \tag{0.1}$$

with $\langle x^2\rangle$ the square mean of position, and k_B the Boltzmann constant. For singly clamped beams (i.e., cantilevers) undergoing harmonic motion and with length l, thickness t and width w, the spring constant is generally well approximated with

$$k \propto \frac{wt^3}{l^3}. \tag{0.2}$$

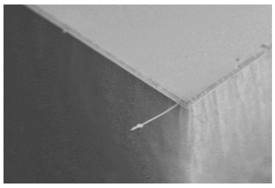
Plugging equation (0.2) into (0.1) and rearranging yields then

$$\langle x^2\rangle \propto k_B T \frac{l^3}{wt^3} \tag{0.3}$$

The amplitude of thermal oscillations is given by $x_{th} = \sqrt{\langle x^2\rangle}$. Let us compare macroscopic and microscopic objects:

A beam of steel with (l,w,t) = (2 m, 10 cm, 5 cm) at 300 K shows thermal fluctuations of $\sim$ 0.2 pm.

x_{th}/t $= 4 \cdot 10^{-12}$.



A nanoscale cantilever with (l,w,t) = (120 μm, 3 μm, 100 nm) at 300 K shows thermal fluctuations of $\sim$8 nm.

x_{th}/t $= 0.08$.

Surely, thermal fluctuations for the steel bar can be neglected. This, however, is not necessarily true for the cantilever. When going to even smaller objects, quantum mechanical fluctuations start to become relevant as well. Generally, the zero point fluctuations (fluctuations of the ground state) are given by:

$$x_{\text{ZPF}} = \sqrt{\frac{\hbar}{2m\omega}} = \sqrt{\frac{\hbar}{2\sqrt{mk}}} \tag{0.4}$$

$$x_{\text{ZPF}} \propto \sqrt{\frac{\hbar l}{wt^2}} \tag{0.5}$$

For the previously encountered beam of steel, $x_{\text{ZPF}} \sim 0.2$ am. For the cantilever, $x_{\text{ZPF}} \sim 0.2$ pm. For a carbon nanotube with a diameter of ~ 1 nm, $x_{\text{ZPF}} \sim 4$ pm. This corresponds to 0.4% of its diameter!

There are several reasons to study nanomechanics. First of all, this field links classical mechanics and statistical mechanics, as demonstrated with the considerations concerning thermal fluctuations above. Then, it also forms a link between classical mechanics and quantum mechanics.[1] And finally, smaller sensors are more sensitive. This is taken advantage of when measuring very small displacements, masses, forces, charges, magnetic moments, etc. $\rightarrow$ keyword atomic force microscopy (AFM).

[1] This is a rather unique property, as many other fields of physics dealing with quantum phenomena cannot be scaled up to show classical behavior. Consider e.g., atomic physics: This field is either treated completely classically with the Lambert-Beer Law of absorption etc., or completely quantum mechanically with quantized energy levels etc. There is no middle ground.

Format and Requirements

Language: English

Prerequisites: Physics III; course-work in solid-state physics and statistical mechanics

Lectures: 10-12 on Wednesdays

Exercise Sessions: TBA

Assignments: Exercises, reading

Final Exam: oral

Grading: 1.0 - 6.0, based on exam and participation

Literature

- *Foundations of Nanomechanics,* A. N. Cleland (Springer, 2003)
- *Fundamentals of Nanomechanical Resonators,* S. Schmid, L. G. Villanueva, M. L. Roukes (Springer 2016)
- *Fundamentals of Statistical and Thermal Physics,* F. Reif (McGraw-Hill, 1965)
- Original papers from *Nature, Science, Physical Review Letters, Applied Physics Letters, Review of Scientific Instruments, Physics Today,* etc.

Website

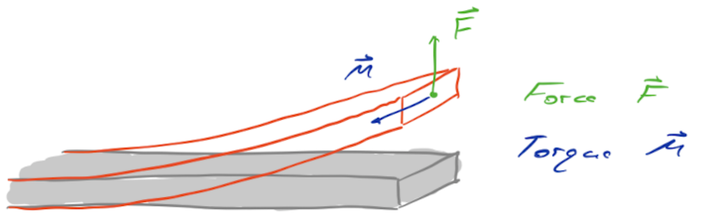
An overview of the lecture, the scripts and slides from the lecture, as well as the problem sheets and reading materials can be found on this website:

`https://poggiolab.unibas.ch/courses/introduction-to-nanomechanics-fall-2020/`

1 Cantilever Basics

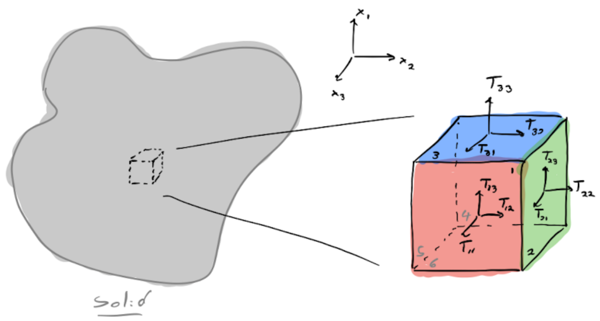
1.1 Nanomechanical resonators

A useful and representative mechanical sensor is a cantilever, i.e., a singly clamped beam. Others include doubly-clamped beams, strings, membranes, etc. Most often, the behavior of these more complex systems can be derived from the considerations of the cantilever as example.



1.2 Stress and Strain (FN p.145)

In order to discuss the motion of a cantilever, the deformation of solids has to be introduced. Consider the forces acting on an infinitesimal cube inside a material:



The force on each surface is given by $\vec{F}_i = \sum_{j=1}^{3} F_{ij}\hat{x}_j$, where $i = 1$ to 6.

We define a vector stress:

$$\underbrace{\vec{t}_i = \frac{\vec{F}_i}{A}}_{\frac{\text{Force}}{\text{area}} = \text{pressure}\left[\frac{N}{m^2}\right]} = \sum_{j=1}^{3} \frac{F_{ij}}{A}\hat{x}_j = \sum_{j=1}^{3} T_{ij}\hat{x}_j \tag{1.1}$$

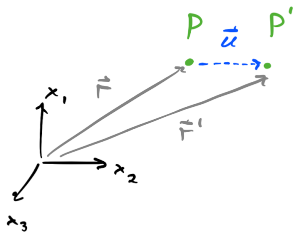
with the **stress tensor $\boldsymbol{T_{ij}}$**.

Since we have an infinitesimal cube in static equilibrium, the forces and torques must be uniform and add to zero:

Fore balance: $\left.\begin{array}{l}\vec{F}_1 = -\vec{F}_4\\ \vec{F}_2 = -\vec{F}_5\\ \vec{F}_3 = -\vec{F}_6\end{array}\right\}$ $\therefore$ We can just consider $i = 1$ to 3.

Torque balance: $\vec{M}_{tot} = 0 \quad \}$ $\therefore$ $T_{ij} = T_{ji}$

Such an applied stress to a solid can result in a deformation, i.e., a **strain**.



The local deformation of a solid is quantified by the relative displacement vector $\vec{u}$ of a point in that solid. Spatial derivatives of this displacement define the **strain tensor**:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.2}$$

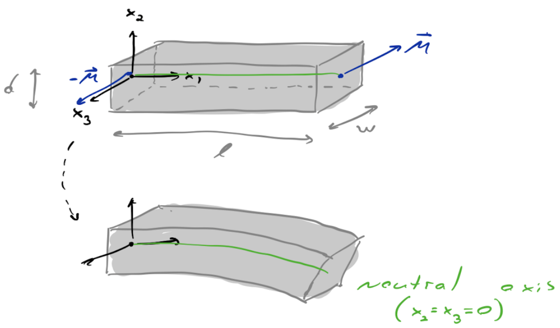
The general **Stress-Strain-Relations** are then given by (FN p.191):

$$T_{ij} = \sum_{k,l} \alpha_{ijk} S_{kl} \tag{1.3}$$

The constants α_{ijk} depend on material parameters:

- E = Young's modulus (stiffness)
- ν = Poisson's ratio (ratio of contraction perpendicular to applied load)

1.3 Example: Bending by Pure Torque (FN p.194)



Static case:

$\vec{M}(x_1 = l) = -M_0 \hat{x}_3$

$\vec{M}(x_1 = 0) = M_0 \hat{x}_3$

One way to apply this type of torque is with the surface stress:

$\vec{t}(0, x_2, x_3) = -t_0 x_2 \hat{x}_1$

$\vec{t}(l, x_2, x_3) = t_0 x_2 \hat{x}_1$

In this case, we need to have:

$$\vec{M}(x_1 = l) = \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{-\frac{w}{2}}^{\frac{w}{2}} \underbrace{\vec{r} \times \vec{t}(l, x_2, x_3)}_{t_0(-x_2^2 \hat{x}_3 + x_2 x_3 \hat{x}_2)} \, \mathrm{d}x_3 \, \mathrm{d}x_2 \tag{1.4}$$

$$= -t_0 x_2^2 \hat{x}_3$$

$$\vec{M}(x_1 = l) = \int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{-\frac{w}{2}}^{\frac{w}{2}} (-t_0 x_2^2 \hat{x}_3) \, \mathrm{d}x_3 \, \mathrm{d}x_2 = -\frac{1}{12} w d^3 t_0 \hat{x}_3 \tag{1.5}$$

Therefore:

$$M_0 = \frac{wd^3}{12} t_0 \tag{1.6}$$

$$t_0 = \frac{12M_0}{wd^3} = \frac{M_0}{I_3}, \qquad \text{with } I_3 = \frac{wd^3}{12} \tag{1.7}$$

The second moments of inertia are defined as

$$I_3 = \int x_2^2 \, \mathrm{d}A \tag{1.8}$$

$$I_2 = \int x_3^2 \, \mathrm{d}A \tag{1.9}$$

In terms of our stress tensor, we have:

$$T_{11} = t_0 x_2 = \frac{M_0}{I_3} x_2 \tag{1.10}$$

All other terms are zero.

By applying the stress-strain relations and boundary conditions, we can then find the resulting deformation (FN p.199):

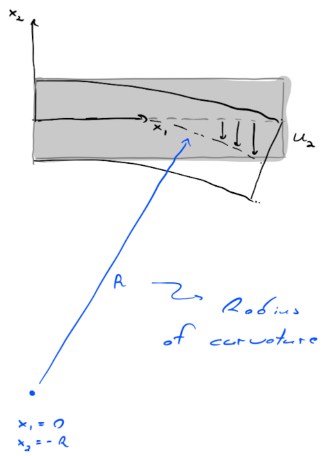
$$u_1 = \frac{M_0}{I_3} \frac{\nu}{E} x_1 x_2 \tag{1.11}$$

$$u_2 = -\frac{M_0}{I_3} \frac{1}{2E} \left(x_1^2 + \nu x_2^2 - \nu x_3^2 \right) \tag{1.12}$$

$$u_3 = -\frac{M_0}{I_3} \frac{\nu}{E} x_2 x_3 \tag{1.13}$$

Along the neutral axis, i.e. $x_2 = x_3 = 0$, we have in the case of $\vec{M} = -M_0 \hat{x}_3$ applied at $x_1 = l$:

$$\boxed{u_2 = -\frac{M_0}{2EI_3} x_1^2} \tag{1.14}$$



Considering the radius of curvature R, the equation of a circle reads:

$$x_1^2 + (R + u_2)^2 = R^2 \tag{1.15}$$

$$u_2 = \sqrt{R^2 - x_1^2} - R \tag{1.16}$$

$$u_2 = R\sqrt{1 - \left(\frac{x_1}{R}\right)^2} - R \tag{1.17}$$

In the case of $x_1 \ll R$, we have:

$$u_2 \approx R\left(1 - \frac{1}{2}\left(\frac{x_1}{R}\right)^2\right) - R \tag{1.18}$$

$$\boxed{u_2 \approx -\frac{x_1^2}{2R}} \tag{1.19}$$

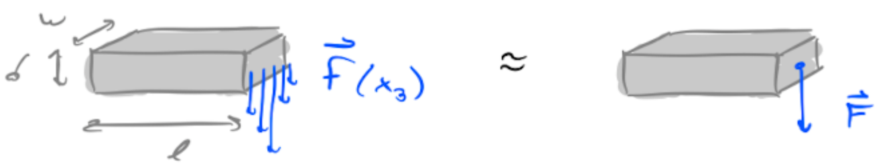
Combining equations (1.14) and (1.19) yields:

$$\boxed{R \approx \frac{EI_3}{M_0}} \tag{1.20}$$

1.4 Euler-Bernoulli Theory of Beams

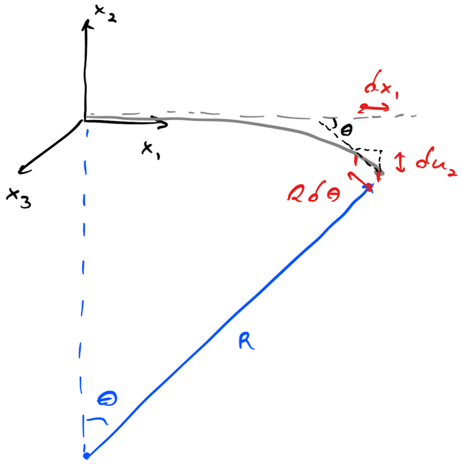
Saint-Venant's Principle:

Given a beam much longer than thick and wide, i.e., $w, d \ll l$, we can consider a force distribution at the end of the beam to be a point-force.



Local Radius of Curvature:

For a slightly bent beam, the radius of curvature R is much larger than its length (no bending: $R \to \infty$). In the slightly-bent beam limit:



$$R \gg l$$
$$R \gg R\Theta$$
$$\Theta \ll 1$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}x_1} = -\tan\Theta \approx -\Theta$$

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}x_1^2} \approx -\frac{\mathrm{d}\Theta}{\mathrm{d}x_1}$$

$$\mathrm{d}x_1 \approx R\,\mathrm{d}\Theta$$

$$\frac{\mathrm{d}^2u_2}{\mathrm{d}x_1^2} \approx -\frac{1}{R} = -\frac{M_0}{EI_3}$$

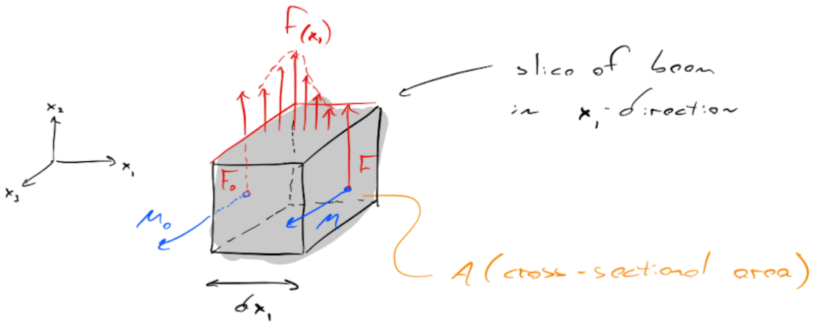
Therefore, in the Euler-Bernoulli Limit:

$$\boxed{\frac{\mathrm{d}^2u_2}{\mathrm{d}x_1^2} \approx -\frac{M_0}{EI_3}} \tag{1.21}$$

2 Cantilever Statics: Beam Bending Formula

2.1 Force & torque balances

To compute the bending of a beam, we consider a slice of the beam along the x_1 direction:



The force balance dictates:

$$\text{Full beam:} \quad F_0 + \int_0^l f(x_1')A\,\mathrm{d}x_1' + F(l) = 0 \tag{2.1}$$

$$\text{Any part of the beam:} \quad F_0 + \int_0^{x_1} f(x_1')A\;\mathrm{d}x_1' + F(x_1) = 0 \tag{2.2}$$

Combining these equations and differentiation leads to:

$$F(x_1) = -F_0 - \int_0^{x_1} f(x_1')A\,\mathrm{d}x_1' \tag{2.3}$$

$$\frac{\mathrm{d}F}{\mathrm{d}x_1} = -f(x_1)\,A \quad \Longrightarrow \boxed{\mathrm{d}F = -f(x_1)A\mathrm{d}x_1} \tag{2.4}$$

The torque balance dictates:

$$\text{Full beam:} \quad M_0 + \int_0^l x_1' f(x_1')A\,\mathrm{d}x_1' + lF(l) + M(l) = 0 \tag{2.5}$$

$$\text{Any part of the beam:} \quad M_0 + \int_0^{x_1'} x_1' f(x_1')A\,\mathrm{d}x_1' + x_1F(x_1) + M(x_1) = 0 \tag{2.6}$$

Again, combining the equations and differentiation leads to:

$$M(x_1) = -M_0 - \int_0^{x_1} x_1' f(x_1')A\,\mathrm{d}x_1' - x_1F(x_1) \tag{2.7}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x_1} = -x_1 f(x_1)A - F(x_1) - x_1\frac{\mathrm{d}F}{\mathrm{d}x_1}(x_1) \tag{2.8}$$

$$\boxed{\mathrm{d}M = -x_1 f(x_1)A\,\mathrm{d}x_1 - F\,\mathrm{d}x_1 - x_1\,\mathrm{d}F} \tag{2.9}$$

Combining equations (2.4) and (2.9):

$$\frac{\mathrm{d}M}{\mathrm{d}x_1} = -x_1 f(x_1)A - F - x_1 \frac{\mathrm{d}F}{\mathrm{d}x_1} \tag{2.10}$$

$$= \cancel{-x_1 f(x_1)A} - F + \cancel{x_1 f(x_1)A} \tag{2.11}$$

$$\tag{2.12}$$

Differentiating one more time and using equation (2.4):

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x_1^2} = f(x_1)A \tag{2.13}$$

Recall equation (1.21):

$$\boxed{\frac{\mathrm{d}^2 u_2}{\mathrm{d}x_1^2} = \frac{M_0}{EI_3}} \tag{2.14}$$

Putting everything together, we arrive at the beam bending formula:

$$\frac{\mathrm{d}^4 u_2}{\mathrm{d}x_1^4} = \frac{1}{EI_3}\frac{\mathrm{d}^2 M}{\mathrm{d}x_1^2} \quad \Longrightarrow \quad \boxed{\frac{\mathrm{d}^4 u_2}{\mathrm{d}x_1^4} = \frac{f(x_1)A}{EI_3}} \tag{2.15}$$

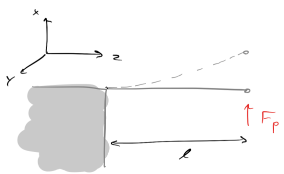
2.2 Examples

Let us use these two (boxed) formulas to solve some simple static examples:

Example 1: Point Force

Consider a singly-clamped beam with a point force applied to its end.

(Force density: $F_p \cdot \frac{\delta(z-l)}{A} \quad \left[\frac{N}{m^3}\right]$)



There are the following boundary conditions (BC):

1. $u_x(0) = 0$: There is no displacement at the clamping point.
2. $\frac{\mathrm{d}u_x}{\mathrm{d}z}(0) = 0$: There is no bending at the clamping point.
3. $\frac{\mathrm{d}^2 u_x}{\mathrm{d}z^2}(l) = 0$: There is no torque applied to the end of the beam.

To solve the problem, we start with equation 2.15, integrate four times and apply the boundary conditions:

$$\frac{\mathrm{d}^4 u_x}{\mathrm{d}z^4} = \frac{f(z)A}{EI_y} = \frac{1}{EI_y}\left[-F_p\delta(z) + F_p\delta(z-l)\right] \tag{2.16}$$

$$\frac{\mathrm{d}^3 u_x}{\mathrm{d}z^3} = -\frac{F_p}{EI_y} \quad \text{for } z < l \tag{2.17}$$

$$\frac{\mathrm{d}^2 u_x}{\mathrm{d}z^2} = -\frac{F_p}{EI_y}z + c_1 \quad , \quad \text{where } c_1 = \frac{F_p l}{EI_y} \quad \text{(from BC 3)} \tag{2.18}$$

$$\frac{\mathrm{d}u_x}{\mathrm{d}z} = -\frac{F_p}{2EI_y}z^2 + \frac{F_p l}{EI_y}z + c_2 \quad , \quad \text{where } c_2 = 0 \quad \text{(from BC 2)} \tag{2.19}$$

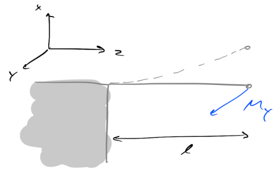
$$u_x = -\frac{F_p}{6EI_y}z^3 + \frac{F_p l}{2EI_y}z^2 + c_3 \quad , \quad \text{where } c_3 = 0 \quad \text{(from BC 1)} \tag{2.20}$$

Finally, we arrive at

$$\boxed{u_x = \frac{F_p}{2EI_y}\left(lz^2 - \frac{z^3}{3}\right)} \tag{2.21}$$

Example 2: Point Torque

Now, consider a singly-clamped beam with a point torque applied to its end.



In this case, the boundary conditions (BC) are:

1. $u_x(0) = 0$: There is no displacement at the clamping point.
2. $\frac{du_x}{dz}(0) = 0$: There is no bending at the clamping point.

To solve this problem we start with equation (2.14), integrate twice and apply the boundary conditions:

$$\frac{d^2u_x}{dz^2} = \frac{M_y}{EI_y} \tag{2.22}$$

$$\frac{du_x}{dz} = \frac{M_y}{EI_y}z + c_1 \quad , \quad \text{where } c_1 = 0 \quad \text{(from BC 2)} \tag{2.23}$$

$$u_x = \frac{M_y}{2EI_y}z^2 + c_2 \quad , \quad \text{where } c_2 = 0 \quad \text{(from BC 1)} \tag{2.24}$$

Therefore, the bending of the beam is given by:

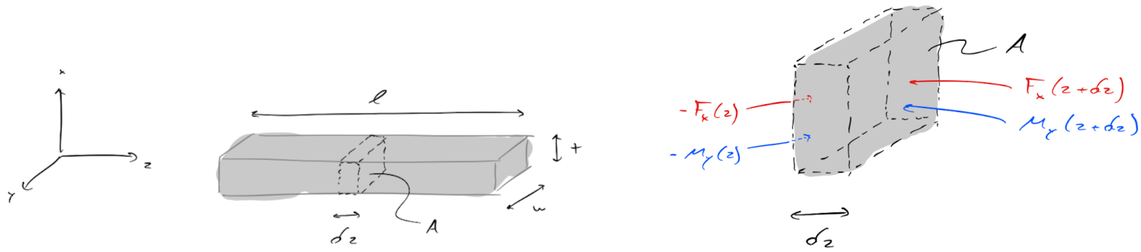
$$\boxed{u_x = \frac{M_y}{2EI_y}z^2} \tag{2.25}$$

3 Cantilever Dynamics

So far, we have treated only static forces and torques, which caused bending of cantilevers. Now, we want to investigate the dynamic behavior of cantilevers, especially under oscillating forces.

3.1 Flexural Vibrations

Let us calculate the dynamic behavior of this beam as the end flexes in the x-direction:



Balance Forces:

$$F_x(z + \mathrm{d}z) - F_x(z) = \underbrace{\rho A \,\mathrm{d}z}_{\text{mass}} \cdot \underbrace{\frac{\mathrm{d}^2 u_x}{\mathrm{d}t^2}}_{\text{acceleration}} \tag{3.1}$$

Balance Torques:

$$M_y(z + \mathrm{d}z) - M_y(z) + F_x(z + \mathrm{d}z)\,\mathrm{d}z = 0 \tag{3.2}$$

Expanding out for small dz around z:

$$\boxed{\begin{aligned} \frac{\partial F_x}{\partial z} &= \rho A \frac{\partial^2 u_x}{\partial t^2} \quad (3.3) \\ \frac{\partial M_y}{\partial z} &= -F_x(z) \quad (3.4) \end{aligned}} \implies \boxed{\frac{\partial^2 M_y}{\partial z^2} = -\rho A \frac{\partial^2 u_x}{\partial t^2}} \quad (3.5)$$

Recall from before: $\frac{\mathrm{d}^2 u_x}{\mathrm{d}z^2} = \frac{M_y}{EI_y}$. Then, it follows:

$$M_y = EI_y \frac{\partial^2 u_x}{\partial z^2} \tag{3.6}$$

$$EI_y \frac{\partial^4 u_x}{\partial z^4} = -\rho A \frac{\partial^2 u_x}{\partial t^2} \tag{3.7}$$

$$\implies \boxed{EI_y \frac{\partial^4 u_x}{\partial z^4} + \rho A \frac{\partial^2 u_x}{\partial t^2} = 0} \tag{3.8}$$

Let us now assume harmonic time dependence for the displacement:

$$u_x(z, t) = u_x(z)\,\mathrm{e}^{-i\omega t} \tag{3.9}$$

$$\implies \frac{\mathrm{d}^4 u_x}{\mathrm{d}z^4} = \left(\frac{\rho A}{EI_y}\right) \omega^2 u_x(z) \qquad \text{Define } \beta \equiv \left(\frac{\rho A}{EI_y}\right)^{\frac{1}{4}} \omega^{\frac{1}{2}} \tag{3.10}$$

The particular solutions for $u_x(z)$ are then of the form:

$$u_x(z) = \mathrm{e}^{\kappa z}\,, \qquad \text{with } \kappa = \pm\beta, \pm i\beta \tag{3.11}$$

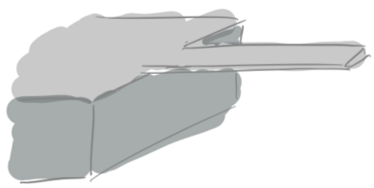
A general solution for $u_x(z)$ is:

$$u_x(z) = A\,\mathrm{e}^{i\beta z} + B\,\mathrm{e}^{-i\beta z} + C\,\mathrm{e}^{\beta z} + D\,\mathrm{e}^{-\beta z} \tag{3.12}$$

or equivalently:

$$u_x(z) = a\cos(\beta z) + b\sin(\beta z) + c\cosh(\beta z) + d\sinh(\beta z) \tag{3.13}$$

With this result, we can now apply boundary conditions to find the motion of a singly clamped beam (i.e., a cantilever):



1. $u_x(0) = 0$ → no displacement at clamp
2. $\frac{\mathrm{d}u_x}{\mathrm{d}z}(0) = 0$ → no bendig at clamp
3. $\frac{\mathrm{d}^2u_x}{\mathrm{d}z^2}(l) = 0$ → no torque at free end
4. $\frac{\mathrm{d}^3u_x}{\mathrm{d}z^3}(l) = 0$ → no net force over full beam

Going through the math, one finds that these conditions require:

$$a = -c \tag{3.14}$$

$$b = -d \tag{3.15}$$

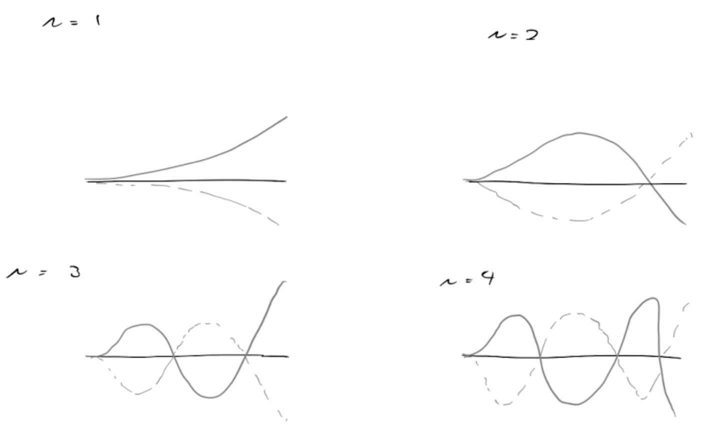
and

$$\underbrace{\cos(\beta l)\cosh(\beta l) + 1 = 0}_{\beta_n l = 1.875,\,4.644,\,7.855,\,10.446,\,\ldots} \tag{3.16}$$

Therefore, complete solutions with the parameter n are given by:

$$\boxed{\begin{array}{c} u_{x,n}(z) = a_n\left[\cos(\beta_n z) - \cosh(\beta_n z)\right] + b_n\left[\sin(\beta_n z) - \sinh(\beta_n z)\right] \\ \text{with} \quad \frac{a_n}{b_n} = -1.362,\ -0.982,\ -1.008,\ -1.000,\ \ldots \end{array}} \tag{3.17}$$

These solutions take the following form:



with angular frequencies

$$\boxed{\omega_n = \sqrt{\frac{EI_y}{\rho A}}\beta_n^2} \tag{3.18}$$

3.2 Zener's Model of an Anelastic Solid (FN p.282)

One model which describes solids in a more realistic way by introducing losses, is the Zener's model. Beginning from a linear stress-strain relation:

$$\text{stress} = \text{Young's Modulus} \cdot \text{strain}$$
$$\iff \quad \sigma = E \cdot \Sigma \tag{3.19}$$

We introduce a time dependence of stress and strain:

$$\sigma + T_\Sigma \frac{\mathrm{d}\sigma}{\mathrm{d}t} = E_R \left(\Sigma + T_\sigma \frac{\mathrm{d}\Sigma}{\mathrm{d}t} \right) \tag{3.20}$$

Now, we consider harmonic motion:

$$\sigma = \sigma_0 \, \mathrm{e}^{i\omega t} \qquad \Sigma = \Sigma_0 \, \mathrm{e}^{i\omega t} \tag{3.21}$$

$$\frac{\sigma_0}{\Sigma_0} = E_R \left[\frac{1 + T_\sigma \, i\omega}{1 + T_\Sigma \, i\omega} \right] \equiv E(\omega) \tag{3.22}$$

This can be expressed differently by introducing following definitions:

$$T \equiv \sqrt{T_\sigma T_\Sigma} \qquad \Delta \equiv \frac{T_\sigma - T_\sigma}{T} \qquad E_{\text{eff}}(\omega) \equiv E_R \frac{1 + \omega^2 T^2}{1 + \omega^2 T_\Sigma^2} \tag{3.23}$$

$$\implies \quad E(\omega) = E_R \left[\frac{1 + \omega^2 T^2}{1 + \omega^2 T_\Sigma^2} + i \, \frac{\omega \; T}{1 + \omega^2 T_\Sigma^2} \Delta \right] \tag{3.24}$$

$$E(\omega) = E_{\text{eff}}(\omega) \left[1 + i \, \frac{\omega T}{1 + \omega^2 T^2} \Delta \right] \tag{3.25}$$

$$E(\omega) = E_{\text{eff}}(\omega) \left[1 + \frac{i}{Q} \right] , \quad \text{with} \ \frac{1}{Q} \equiv \frac{\omega T}{1 + \omega^2 T^2} \Delta \tag{3.26}$$

Inserting this back into the result from Euler Bernoulli (3.8):

$$E I_y \frac{\partial^4 u_x}{\partial z^4} + \rho A \frac{\partial^2 u_x}{\partial t^2} = 0 \tag{3.27}$$

$$\frac{\partial^4 u_x}{\partial z^4} (z) = \underbrace{\left[\frac{\rho A}{E_{\text{eff}} \left(1 + \frac{i}{Q} \right) I_y} \right]}_{\beta^4} \omega^2 \, u_x(z) \tag{3.28}$$

This results in new angular frequencies:

$$\boxed{\begin{aligned} \omega_n' &= \sqrt{\frac{E_{\text{eff}} I_y}{\rho A}}\,\beta_n^2 \left(1 + \frac{i}{2Q}\right) && (3.29)\\ \omega_n' &= \left(1 + \frac{i}{2Q}\right)\omega_n \qquad \text{for } Q \gg 1 && (3.30) \end{aligned}}$$

3.3 Cantilevers as Harmonic Oscillators

So far, we have not considered any external forces on our cantilever. Let us add a driving field, and equation (3.8) becomes:

$$EI_y \frac{\partial^4 u_x}{\partial z^4} + \rho A \frac{\partial^2 u_x}{\partial t^2} = \underbrace{f(t)}_{\left[\frac{\text{Force}}{\text{length}}\right]} \tag{3.31}$$

If we Fourier decompose $f(t)$, we can do the same for u_x, considering:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega)\, \mathrm{e}^{i\omega t}\, \mathrm{d}\omega \tag{3.32}$$

$$u_x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}_x(\omega)\, \mathrm{e}^{i\omega t}\, \mathrm{d}\omega \tag{3.33}$$

Plugging in these expressions and canceling out both the time dependence and integrals from the equations, we have:

$$EI_y \frac{\partial^4 \hat{U}_x}{\partial z^4} - \omega^2 \rho A \hat{U}_x = \hat{F}(\omega) \tag{3.34}$$

$\hat{U}_x$ can be written in terms of the eigenfunctions of the beam:

$$\hat{U}_x = \sum_{n=1}^{\infty} a_n u_{xn}\,, \qquad \text{where} \int_0^{\infty} u_{xn} u_{xm}\, \mathrm{d}z = l^3 \delta_{mn} \quad \text{(orthogonality)} \tag{3.35}$$

Then:

$$EI_y \sum_{n=1}^{\infty} a_n \underbrace{\frac{\partial^4 u_{xn}}{\partial z^4}} - \omega^2 \rho A \sum_{n=1}^{\infty} a_n u_{xn} = \hat{F}(\omega) \tag{3.36}$$

$$\text{From before: } \frac{\partial^4 u_{xn}}{\partial z^4} = \left(\frac{\rho A}{EI_y}\right) \omega_n^2 u_{xn}$$

$$\rho A \sum_{n=1}^{\infty} a_n \omega_n'^2\, u_{xn} - \rho A \omega^2 \sum_{n=1}^{\infty} a_n u_{xn} = \hat{F}(\omega) \tag{3.37}$$

Integrating through with u_{xn}:

$$\rho A \left[\sum_{n=1}^{\infty} a_n \omega_n'^2 \underbrace{\int_0^l u_{xn} u_{xm} \, \mathrm{d}z}_{l^3 \delta_{nm}} - \omega^2 \sum_{n=1}^{\infty} a_n \underbrace{\int_0^l u_{xm} u_{xn} \, \mathrm{d}z}_{l^3 \delta_{nm}} \right] = \int_0^l u_{xn} \hat{F}(\omega) \, \mathrm{d}z \tag{3.38}$$

$$\rho A l^3 a_n \left(\omega_n'^2 - \omega^2 \right) = \int_0^l u_{xn} \hat{F}(\omega) \, \mathrm{d}z \tag{3.39}$$

$$\implies \quad a_n = \underbrace{\frac{1}{\rho A l^3}}_{m \cdot l^2} \frac{1}{\omega_n'^2 - \omega^2} \int_0^l u_{xn} \hat{F}(\omega) \, \mathrm{d}z \tag{3.40}$$

recall from earlier: $\omega_n' = \left(1 + \frac{i}{2Q} \right) \omega_n$

In the limit of high Q (i.e., small dissipation):

$$\boxed{a_n = \frac{1}{ml^2} \int_0^l u_{xn} \hat{F}(\omega) \, \mathrm{d}z \left(\frac{1}{\omega_n^2 - \omega^2 + i \frac{\omega_n^2}{Q}} \right)} \tag{3.41}$$

The response of each mode n - as we will see - looks a lot like the response of a simple driven damped harmonic oscillator. Let us now specify that our driving force $f(t)$ is a point force applied at $z = l$:

$$\implies \quad \hat{F}(\omega) = \hat{F}_p(\omega) \, \delta(z - l) \tag{3.42}$$

$$\int_0^l u_{xn} \hat{F}(\omega) \, \mathrm{d}z = u_{xn}(l) \, \hat{F}_p(\omega) \tag{3.43}$$

If we now consider the response of the fundamental mode, $n = 1$, for our normalization we know that $u_{x1}(l) \approx 2l$.

$$\implies \quad a_1 = \frac{1}{ml^2} \, 2l \hat{F}_p(\omega) \, \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}} \tag{3.44}$$

$$a_1 = \frac{2 \hat{F}_p(\omega)}{ml} \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}} \tag{3.45}$$

Driven displacement of the $n = 1$ mode at the end of the cantilever:

$$a_1 u_{x1}(l) = \frac{4 \hat{F}_p(\omega)}{m} \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}} \tag{3.46}$$

If ω is close to ω_1 and no other mode resonates, then we can say that the Fourier component of the cantilever's end deflection is:

$$\hat{x}_{\text{end}}(\omega) = a_1 u_{x1}(l) = \frac{4 \hat{F}_p(\omega)}{m} \frac{1}{\omega_1^2 - \omega^2 + i \frac{\omega_1^2}{Q}} \tag{3.47}$$

$$\boxed{\hat{x}_{\text{end}}(\omega) = \frac{\hat{F}_p(\omega)}{m_{\text{eff}}} \frac{1}{\omega_1^2 - \omega^2 + i\frac{\omega_1^2}{Q}}} \tag{3.48}$$

$$\text{with } m_{\text{eff}} = \frac{m}{4} \tag{3.49}$$

This expression can be rewritten in terms of a mechanical susceptibility $\chi_m(\omega)$:

$$\hat{x}_{\text{end}}(\omega) = \hat{F}_p(\omega)\, \chi_m(\omega) \tag{3.50}$$

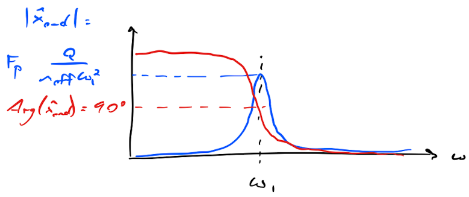
$$\text{with } \chi_m(\omega) = \frac{1}{m_{\text{eff}}} \frac{1}{\omega_1^2 - \omega^2 + i\frac{\omega_1^2}{Q}} \tag{3.51}$$

At driving forces matching the resonance frequency ω_1:

$$\chi_m(\omega_1) = -i \cdot \frac{Q}{m_{\text{eff}}\, \omega_1^2} \tag{3.52}$$

$$\text{-90° phase shift} \qquad \text{response multiplied by Q} \tag{3.53}$$

The cantilever behaves as a resonator, responding only to forces at frequency ω_1, within a bandwidth proportional to $\frac{1}{Q}$.



Recall what we derived for the static case: for a point force applied to the end, a cantilever bends:

$$u_x(z) = \frac{F_p}{2EI_y}\left(lz^2 - \frac{z^3}{3}\right) \tag{3.54}$$

$$u_x(l) = F_p \frac{l^3}{3EI_y} \qquad I_y = \frac{wt^3}{12} \tag{3.55}$$

$$u_x(l) = F_p\left(\frac{4l^3}{Ewt^3}\right) \tag{3.56}$$

$$\Longrightarrow \quad F_p = k_s \cdot u_x(l) \tag{3.57}$$

$$\boxed{\textbf{static spring constant: } k_s = \frac{Ewt^3}{4l^3}} \tag{3.58}$$

Compare to the dynamic case:

$$\hat{F}_p(\omega) = \frac{1}{\chi_m(\omega)}\, \hat{x}_{\text{end}}(\omega) \tag{3.59}$$

$$k_D = \frac{1}{\chi(\omega)} \tag{3.60}$$

On resonance:

$$k_D = \frac{1}{\chi(\omega_1)} = i\,\frac{m_{\text{eff}}\omega_1^2}{Q} \qquad \text{recall: } \omega_1 = \sqrt{\frac{EI_y}{\rho A}}\beta_n^2 \tag{3.61}$$

$$= i\,\frac{\rho wtl}{4Q}\,\frac{EI_y}{\rho wt}\,\frac{1.875^4}{l^4} \tag{3.62}$$

$$= \frac{i}{Q}\,\frac{Ewt^3}{48l^3}\,\underbrace{1.875^4}_{12.36} \tag{3.63}$$

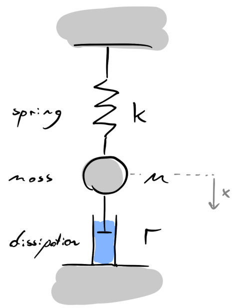
$$\approx \frac{i}{Q}\,\frac{Ewt^3}{4l^3} = \frac{i}{Q}\,k_s \tag{3.64}$$

$$\boxed{\textbf{dynamic spring constant: } k_D \approx \frac{i}{Q}k_s} \tag{3.65}$$

We see that the dynamic spring constant of a cantilever driven at resonance frequency of the first mode is smaller by a factor Q, i.e., the cantilever is softer in this case.

3.4 Simple Harmonic Oscillator

Let us for a moment consider the simple harmonic oscillator:



Equation of motion:

$$m\ddot{x} + \Gamma\dot{x} + kx = F(t) \tag{3.66}$$

Let us define the Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega \tag{3.67}$$

If we take the Fourier transform of the equation of motion, we get:

$$-m\omega^2\hat{x}(\omega) + i\,\Gamma\omega\hat{x}(\omega) + kx(\omega) = \hat{F}(\omega) \tag{3.68}$$

$$\hat{x}(\omega) = \frac{\hat{F}(\omega)}{m}\,\frac{1}{\frac{k}{m} - \omega^2 + i\Gamma\omega} \tag{3.69}$$

$$\tag{3.70}$$

We use these definitions:

$$k = m\omega_0^2 \qquad \Gamma = \frac{m\omega_0}{Q} \tag{3.71}$$

to obtain

$$\boxed{\hat{x}(\omega) = \frac{\hat{F}(\omega)}{m}\,\frac{1}{\omega_0^2 - \omega^2 + i\frac{\omega_0\omega}{Q}}} \tag{3.72}$$

This expression is very similar to that of a driven, damped beam. Here, again, we can define a susceptibility:

$$\hat{x}(\omega) = \hat{F}(\omega)\,\chi_m(\omega) \tag{3.73}$$

$$\text{with } \chi_m(\omega) = \frac{1}{m}\,\frac{1}{\omega_0^2 - \omega^2 + i\frac{\omega_0\omega}{Q}} \tag{3.74}$$

On resonance, the harmonic oscillator reacts:

$$\chi_m(\omega_0) = -i \cdot \frac{Q}{m\omega_0^2} \tag{3.75}$$

$$\text{-90° phase shift} \qquad \text{response multiplied by Q} \tag{3.76}$$

Similarly, the dynamic spring constant is softened by Q:

$$k_D = \frac{1}{\chi_m(\omega)} = i\,\frac{m\omega_0^2}{Q} \tag{3.77}$$

$$k_D = i\,\frac{k}{Q} \tag{3.78}$$

So, the beam and the harmonic oscillator behave almost the same! There are two main differences:

1. m vs. $m_{\text{eff}} = \frac{m}{4}$: when the beam is oscillating, not the whole mass is involved in the movement.
2. $i\,\frac{\omega_0\omega}{Q}$ vs $i\,\frac{\omega_1^2}{Q}$: this difference is only relevant off resonance and for systems with low Q.

4 Dissipation and Noise

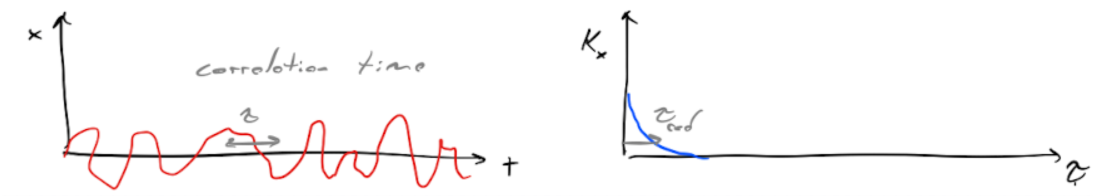
4.1 Power Spectral Density

We now know how beams - and by extension other continuous objects like strings, membranes, and rods - behave under both static and dynamic forces. We find that they act as transducers and as resonators with geometry and losses determining their properties. We have written transfer functions and can solve for their response to time-varying forces. Now, we must consider the effects of fluctuations.

First, however, we have to discuss how to quantify fluctuations. For this, we begin by defining the correlation function:

$$K_x(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{x(t)\, x^*(t-\tau)}{T}\, \mathrm{d}t \tag{4.1}$$

The above function measures how well correlated fluctuations are after a time τ.



If we write the integrand in terms of the Fourier Transforms of $x(t)$ and $x^*(t-\tau)$:

$$K_x(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\hat{x}(\omega)\, \hat{x}^*(\omega')}{T}\, \mathrm{e}^{i(\omega-\omega')t}\, \mathrm{e}^{i\omega'\tau}\, \mathrm{d}\omega\, \mathrm{d}\omega'\, \mathrm{d}t \tag{4.2}$$

$$\text{Recall: } \delta(\omega-\omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i(\omega-\omega')t} \mathrm{d}t \tag{4.3}$$

$$K_x(\tau) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\hat{x}(\omega)\, \hat{x}^*(\omega')}{T}\, \delta(\omega-\omega')\, \mathrm{e}^{i\omega'\tau}\, \mathrm{d}\omega\, \mathrm{d}\omega' \tag{4.4}$$

$$K_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\lim_{T\to\infty} \frac{\hat{x}(\omega)\, \hat{x}^*(\omega)}{T}}\, \mathrm{e}^{i\omega\tau}\, \mathrm{d}\omega \tag{4.5}$$

$$\boxed{S_x(\omega) = \lim_{T\to\infty} \frac{\hat{x}(\omega)\, \hat{x}^*(\omega)}{T}} \quad \textbf{Power Spectral Density} \tag{4.6}$$

$$\boxed{K_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\, \mathrm{e}^{i\omega\tau}\, \mathrm{d}\omega} \quad \textbf{Correlation Function} \tag{4.7}$$

Correlation function and power spectral density (PSD) are Fourier Transform pairs, and the PSD has units of $[m^2/\mathrm{Hz}]$.

This Fourier relation also means that:

$$K_x(0) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{x(t)\,x^*(t)}{T}\,\mathrm{d}t = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{|x(t)|^2}{T}\,\mathrm{d}t = \langle x^2 \rangle \tag{4.8}$$

$$\boxed{K_x(0) = \langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega} \tag{4.9}$$

The root-mean-square fluctuations can be expressed as:

$$x_{\mathrm{rms}} = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega} \tag{4.10}$$

What is $S_x(\omega)$ for a beam? Well, we take the definition of the PSD and express it in terms of the Fourier Transforms:

Beam:

$$S_{x,\mathrm{end}}(\omega) = \lim_{T\to\infty} \frac{\hat{x}_{\mathrm{end}}(\omega)\,\hat{x}^*_{\mathrm{end}}(\omega)}{T} \tag{4.11}$$

$$S_{x,\mathrm{end}}(\omega) = \lim_{T\to\infty} \frac{F_p(\omega)\,F^*_p(\omega)}{T} \frac{1}{m^2_{\mathrm{eff}}} \frac{1}{(\omega_1^2 - \omega^2)^2 + \frac{\omega_1^4}{Q^2}} \tag{4.12}$$

$$\boxed{S_{x,\mathrm{end}}(\omega) = \frac{S_{F_p(\omega)}}{m^2_{\mathrm{eff}}} \frac{1}{(\omega_1^2 - \omega^2)^2 + \frac{\omega_1^4}{Q^2}}} \tag{4.13}$$

Similarly, for the **harmonic oscillator:**

$$S_x(\omega) = \lim_{T\to\infty} \frac{\hat{x}(\omega)\,\hat{x}^*(\omega)}{T} \tag{4.14}$$

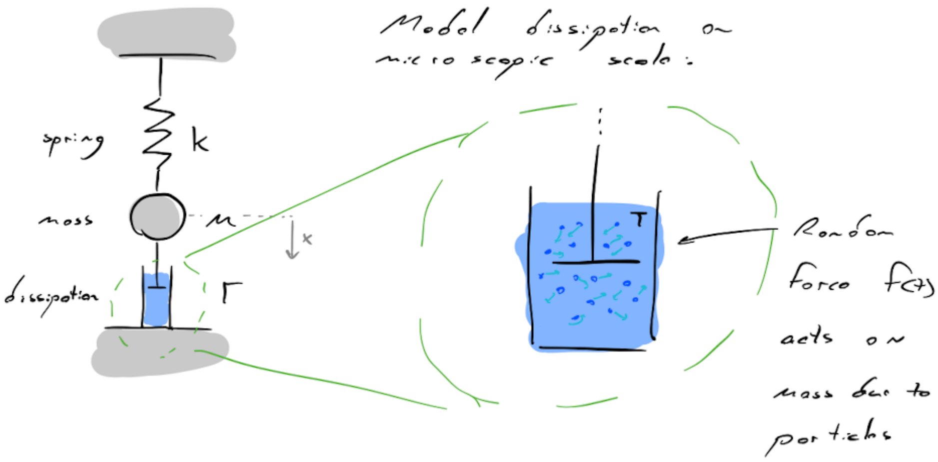
$$\boxed{S_x(\omega) = \frac{S_{F(\omega)}}{m^2} \frac{1}{(\omega_0^2 - \omega^2)^2 + \frac{\omega_0^2\omega^2}{Q^2}}} \tag{4.15}$$

We see that fluctuations in force (S_F) are transduced into fluctuations in displacement (S_x).

Near resonance, i.e., $\omega_0 = \omega_1 \approx \omega$, the beam and the harmonic oscillator respond the same to fluctuations and drives. From now on we will use this as an excuse to approximate the modes of beams, membranes etc. as simple harmonic oscillators.

4.2 Fluctuation - Dissipation

In order to understand the motion of our resonator, we have to understand the force fluctuations that drive it: $S_F(\omega)$. Those, it turns out, are tightly related to the dissipation Γ, which the resonator experiences. Let us consider the harmonic oscillator:



We can re-write the equation of motion to include the random forces:

$$m\ddot{x} + kx = \underbrace{F(t)}_{\text{external driving force}} + \underbrace{f(t)}_{\text{force from microscopic collisions}} \tag{4.16}$$

$$m\ddot{x} = \underbrace{F(t) - kx}_{\substack{\mathscr{F}(t) \\ \text{slowly varying} \\ \text{external forces}}} + \underbrace{f(t)}_{\substack{\text{random forces due to microscopic} \\ \text{collisions with heat reservoir at T}}} \tag{4.17}$$

We cannot know $f(t)$ fully, only statistically. $f(t)$ has a correlation time τ^* which is very short ($\sim 10^{-13}\,s$ for a typical liquid). Consider a macroscopically short time τ such that $\tau \gg \tau^*$:

$$m\left(v(t+\tau) - v(t)\right) = \mathscr{F}(t)\,\tau + \int\limits_t^{t+\tau} f(t')\,\mathrm{d}t' \tag{4.18}$$

Taking an average over the ensemble:

$$m\left\langle v(t+\tau) - v(t)\right\rangle = \mathscr{F}(t)\,\tau + \int\limits_t^{t+\tau} \langle f(t')\rangle\,\mathrm{d}t' \tag{4.19}$$

Let us consider our small system A within a larger heat bath B at temperature T. The probability of A being in some state r, W_r, is proportional to the corresponding number of states available to B, Ω. At time t:

$$W_r(t) \propto \Omega(E') \quad \text{, with } E' \text{ the energy of both systems} \tag{4.20}$$

After some time $\tau' > \tau^*$, so that every accessible state is equally likely:

$$W_r(t+\tau') \propto \Omega(E' + \Delta E') \quad \text{, with } \Delta E' \text{ the energy increase of both systems} \tag{4.21}$$

From statistical mechanics, we know that:

$$\frac{W_r(t+\tau')}{W_r(t)} = \frac{\Omega(E'+\Delta E')}{\Omega(E')} = \exp\left[\frac{\Delta E'}{k_B T}\right] \tag{4.22}$$

In other words, the probability that A is found in a given state r at some later time is increased, if more energy becomes available to B (heat reservoir).

$$W_r(t+\tau') = W_r(t)\,\exp\left[\frac{\Delta E'}{k_B T}\right] \approx W_r(t)\,\left(1+\frac{\Delta E'}{k_B T}\right) \quad \text{(for small changes)} \tag{4.23}$$

$$\langle f(t+\tau')\rangle = \sum_r W_r(t+\tau')\,f_r = \sum_r W_r(t)\,\left(1+\frac{\Delta E'}{k_B T}\right) f_r \tag{4.24}$$

$$\langle f(\underbrace{t+\tau'}_{t'})\rangle = \underbrace{\langle f(t)\rangle}_{0} + \frac{1}{k_B T}\,\left\langle \Delta E' f(t')\right\rangle \quad \text{(zero mean force} \to \text{random)} \tag{4.25}$$

$$\langle f(t')\rangle = \frac{1}{k_B T}\,\left\langle \Delta E' f(t')\right\rangle \tag{4.26}$$

The energy difference $\Delta E'$ can be expressed as:

$$\Delta E' = -\int_t^{t'} v(t'')\,f(t'')\,\mathrm{d}t'' \approx -v(t)\int_t^{t'} f(t'')\,\mathrm{d}t'' \tag{4.27}$$

Using this expression yields

$$\langle f(t')\rangle = \frac{1}{k_B T}\,\left\langle -v(t)\int_t^{t'} f(t'')\,\mathrm{d}t''\,f(t)\right\rangle \tag{4.28}$$

$$= -\frac{\langle v(t)\rangle}{k_B T}\int_t^{t'} \left\langle f(t'')\,f(t)\right\rangle \mathrm{d}t'' \tag{4.29}$$

$$= -\frac{\langle v(t)\rangle}{k_B T}\int_0^{t-t'} \left\langle f(t)\,f(t-s)\right\rangle \mathrm{d}s \qquad s=t-t''\,,\ \mathrm{d}s=-\mathrm{d}t'' \tag{4.30}$$

This expressions occurs integrated in equation (4.19), so let us compute this integral:

$$\int_t^{t+\tau} \langle f(t')\rangle\,\mathrm{d}t' = -\frac{\langle v(t)\rangle}{k_B T}\int_t^{t+\tau}\mathrm{d}t'\int_{t-t'}^{0}\mathrm{d}s\ \underbrace{\left\langle f(t)\,f(t-s)\right\rangle}_{\substack{K_f(s):\\ \text{correlation function}}}$$

One aside about the correlation function: $K_f(s)$ is independent of t. Further, it is symmetric:

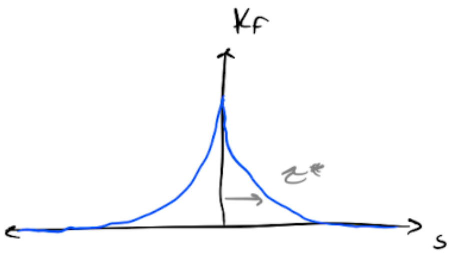
$$K_f(s) = \left\langle f(t)\,f(t-s)\right\rangle = \left\langle f(t_1)\,f(t_1-s)\right\rangle \tag{4.31}$$

$$\text{if } t_1 = t+s:$$

$$K_f(s) = \left\langle f(t)\,f(t-s)\right\rangle = \left\langle f(t+s)\,f(t)\right\rangle \tag{4.32}$$

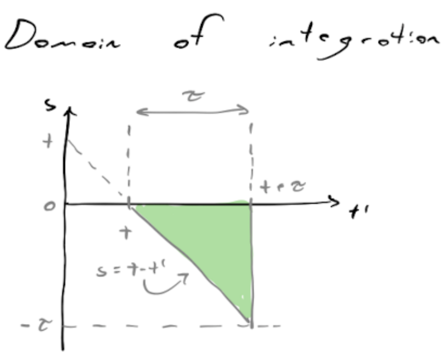
$$\Longrightarrow K_f(s) = K_f(-s) \tag{4.33}$$

Also, it drops to zero for $s \sim \tau^*$:



Let us continue with equations (4.19) and (4.31):

$$m\,\langle v(t+\tau) - v(t)\rangle = \mathscr{F}(t)\,\tau - \frac{\langle v(t)\rangle}{k_B T} \underbrace{\int_t^{t+\tau} \mathrm{d}t' \int_{t-t'}^{0} \mathrm{d}s\, K_s(f)}_{\mathrm{I}} \tag{4.34}$$



$$I = \int_t^{t+\tau} \mathrm{d}t' \int_{t-t'}^{0} \mathrm{d}s\, K_f(s) \tag{4.35}$$

$$= \int_{-\tau}^{0} \mathrm{d}s \int_{t-s}^{t+\tau} \mathrm{d}t'\, K_f(s) \tag{4.36}$$

$$= \int_{-\tau}^{0} \mathrm{d}s\, (\tau + s)\, K_f(s) \tag{4.37}$$

Recall: $\tau \gg \tau^*$ and $K_f(s) \to 0$ for $|s| \gg \tau^*$

$$\implies I \approx \tau \int_{-\infty}^{0} \mathrm{d}s\, K_f(s) = \frac{\tau}{2} \int_{-\infty}^{\infty} \mathrm{d}s\, K_f(s) \qquad K_f(s) \text{ is symmetric} \tag{4.38}$$

Therefore, equation (4.19) can be expressed as:

$$\boxed{m\,\big\langle v(t+\tau) - v(t)\big\rangle = \mathscr{F}(t)\,\tau - \frac{\langle v(t)\rangle}{2k_B T}\,\tau \int_{-\infty}^{\infty} K_f(s)\,\mathrm{d}s} \tag{4.39}$$

Since $\langle v(t)\rangle$ varies slowly over τ:

$$m\,\frac{\mathrm{d}\langle v(t)\rangle}{\mathrm{d}t} = m\,\frac{\langle v(t+\tau)\rangle - \langle v(t)\rangle}{\tau} \tag{4.40}$$

$$\implies m\,\frac{\mathrm{d}\langle v(t)\rangle}{\mathrm{d}t} = \underbrace{\mathscr{F}(t)}_{\text{Recall: } \mathscr{F}(t) = F(t) - kx} - \frac{\langle v(t)\rangle}{2k_B T} \int_{-\infty}^{\infty} K_f(s)\,\mathrm{d}s \tag{4.41}$$

Macroscopically, we have:

$$m\ddot{x} = F(t) - kx - \frac{\dot{x}}{2k_BT}\int_{-\infty}^{\infty} K_f(s)\,\mathrm{d}s \tag{4.42}$$

$$\boxed{m\ddot{x} + \left[\frac{1}{2k_BT}\int_{-\infty}^{\infty} K_f(s)\,\mathrm{d}s\right]\dot{x} + kx = F(t)} \tag{4.43}$$

We can identify the term in square brackets to be the dissipation Γ ! The connection between fluctuating forces $f(t)$ and dissipation Γ is:

$$\Gamma = \frac{1}{2k_BT}\int_{-\infty}^{\infty} K_f(s)\,\mathrm{d}s \tag{4.44}$$

Since the fluctuating forces are nearly uncorrelated on macroscopic timescales ($\tau^* \ll \tau$), we can approximate:

$$K_f(s) \approx f_0^2\delta(s) \tag{4.45}$$

$$\implies \quad \Gamma = \frac{1}{2k_BT}f_0^2 \qquad \implies \quad f_0^2 = 2k_BT\,\Gamma \tag{4.46}$$

$$\implies \boxed{K_f(s) = 2k_BT\,\Gamma\delta(s)} \tag{4.47}$$

In this way we retrieve the original equation of motion from (4.43):

$$m\ddot{x} + \Gamma\dot{x} + kx = F(t) \tag{4.48}$$

This also allows us to find the power spectral density (PSD) of the fluctuating forces $f(t)$:

$$S_f(\omega) = \int_{-\infty}^{\infty} K_f(s)\,\mathrm{e}^{-i\omega s}\,\mathrm{d}s \tag{4.49}$$

$$\boxed{S_f(\omega) = 2k_BT\,\Gamma} \qquad \text{Double-sided:} \ -\infty < \omega < \infty \tag{4.50}$$

Notice that this spectral density is constant in ω, i.e., white!

For real-valued signals $S(\omega)$ is even ($S(\omega) = S(-\omega)$. Therefore, it is sometimes useful to define a single-sided PSD for only positive ω:

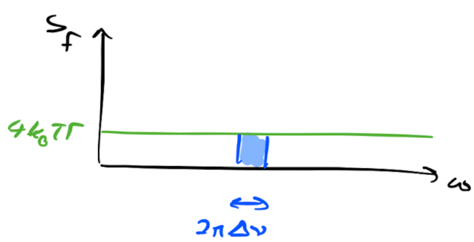
$$\bar{S}(\omega) = S(\omega) + S(-\omega) \tag{4.51}$$

Here:

$$\boxed{\bar{S}_f(\omega) = 4k_BT\,\Gamma} \qquad \text{Single-sided:} \ 0 < \omega < \infty \tag{4.52}$$

This can be translated into a thermal force, which simultaneously determines the minimum detectable force in an experiment with measurement bandwidth $\Delta\nu$:

$$F_{\min} = \sqrt{4k_BT\,\Gamma\Delta\nu} \tag{4.53}$$



Size of thermal displacement fluctuations:

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega \tag{4.54}$$

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_f(\omega)}{m^2} \left(\frac{1}{(\omega_0^2 - \omega^2)^2 + \frac{\omega_0^2 \omega^2}{Q^2}} \right) \mathrm{d}\omega \tag{4.55}$$

$$\langle x^2 \rangle = \frac{k_B T\,\Gamma}{\pi m^2} \underbrace{\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{(\omega_0^2 - \omega^2)^2 + \frac{\Gamma^2 \omega^2}{m^2}}}_{\frac{\pi m}{\omega_0^2 \Gamma}} \tag{4.56}$$

$$\langle x^2 \rangle = \frac{k_B T\,\Gamma}{\pi m^2}\,\frac{\pi m}{\omega_0^2 \Gamma} = \frac{k_B T}{m \omega_0^2} = \frac{k_B T}{k} \tag{4.57}$$

$$\tag{4.58}$$

This can be expressed as

$$\boxed{\frac{1}{2} k \langle x^2 \rangle = \frac{1}{2} k_B T} \tag{4.59}$$

We retrieve the equipartition theorem!

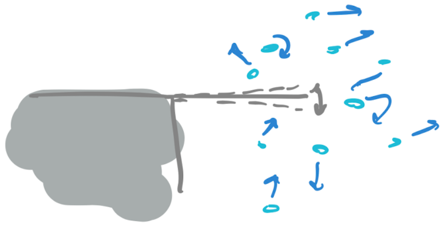
5 Nanomechanical Measurements

5.1 Review and Analogy to Electronics

A quick review:

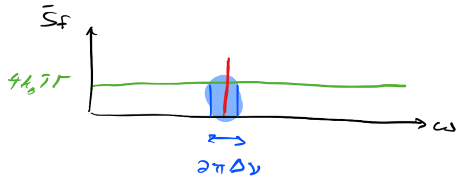
The Langevin Equation for a nanomechanical resonator reads

$$m\ddot{x} + \Gamma\dot{x} + kx = F(t) \tag{5.1}$$



This system has a (single-sided) power spectral density of thermal forces:

$$\boxed{\bar{S}_f(\omega) = 4k_BT\Gamma} \tag{5.2}$$



These thermal forces set a limit to minimum measurable forces:

$$\boxed{F_{\min} = \sqrt{4k_BT\Gamma\Delta\nu}} \tag{5.3}$$

These random force fluctuations drive random displacement fluctuations with a specific spectrum:

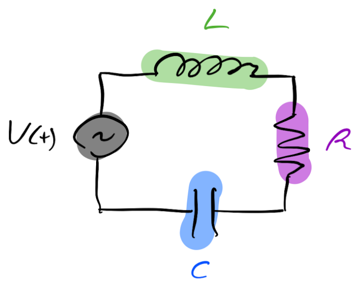
$$\boxed{S_x(\omega) = \frac{2k_BT\Gamma}{m^2}\frac{1}{(\omega_0^2-\omega^2)^2+\frac{\Gamma^2\omega^2}{m^2}}} \tag{5.4}$$

The Langevin equation, which we have been discussing is not the only one. Take for example the standard description of a linear electronic circuit:

$$-L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t) \tag{5.5}$$

$$V_L = -L\frac{\mathrm{d}I}{\mathrm{d}t} \qquad V_R = RI \qquad V_C = \frac{Q}{C}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \ddot{Q} \qquad I = \dot{Q}$$



The PSD of thermal voltage fluctuations is proportional to T and R:

$$\bar{S}_V(\omega) = 4k_BTR \qquad \text{Johnson Noise} \tag{5.6}$$

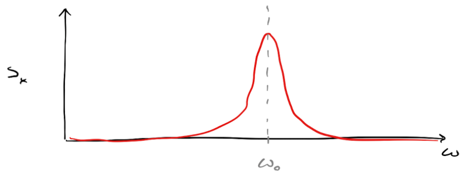
$$V_{\min} = \sqrt{4k_BTR\Delta\nu} \qquad \text{Johnson voltage noise in bandwidth } \Delta\nu \tag{5.7}$$

5.2 Dissipation-induced Amplitude Noise

As we have seen before, the power spectral density of the displacement noise caused by thermal forces is descried by

$$\boxed{S_x(\omega) = \frac{2k_BT\Gamma}{m^2}\frac{1}{(\omega_0^2-\omega^2)^2+\frac{\Gamma^2\omega^2}{m^2}}} \tag{5.8}$$

So even with a white force PSD (i.e., $S_f(\omega) = 2k_BT\Gamma$), the displacement PSD has "color" in the sense that it has a frequency dependence imposed by the mechanical resonator



On resonance:

$$S_x(\omega_0) = \frac{2k_BT\Gamma}{m^2}\frac{m^2}{\Gamma^2\omega_0^2} \tag{5.9}$$

$$S_x(\omega_0) = \frac{2k_BT}{\Gamma\omega_0^2} \tag{5.10}$$

For practical matters, let us consider the single-sided PSD (i.e., $0 < \omega < \infty$) :

$$\bar{S}_x(\omega) = \frac{4k_BT\Gamma}{m^2}\frac{1}{(\omega_0^2-\omega^2)^2+\frac{\Gamma^2\omega^2}{m^2}} \tag{5.11}$$

$$\bar{S}_x(\omega_0) = \frac{4k_BT}{\Gamma\omega_0^2} = \frac{4k_BTQ}{m\omega_0^3} \qquad \left[\frac{m^2}{\text{Hz}}\right] \tag{5.12}$$

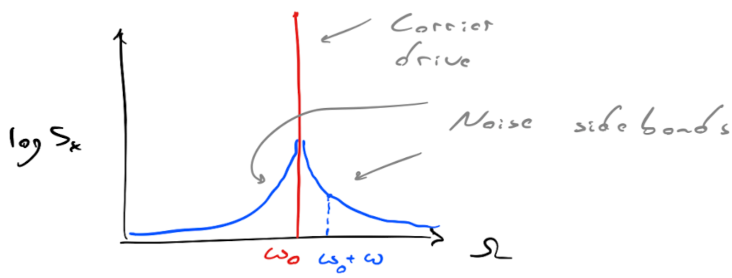
$$x_{\min} = \frac{2}{\omega_0}\sqrt{\frac{k_BT}{\Gamma}\Delta\nu} \qquad \left[\frac{m}{\sqrt{\text{Hz}}}\right] \tag{5.13}$$

$x_{\min}$ is the displacement due to thermal forces and therefore the minimum detectable displacement.

5.3 Dissipation-induced Phase Noise

Thermal fluctuations will not only induce displacement noise (amplitude noise), but also phase noise. This is particularly important for time-keeping or for frequency-based measurements, as we will see.

Take a resonator that is driven by a carrier signal near its resonance frequency ω_0 <u>and</u> dissipation induced thermal force noise: $S_f(\omega) = 2k_BT\Gamma$.



Now we could represent the carrier at ω_0 and a single spectral component of the noise at ω as:

$$x(t) = x_0 \sin(\omega_0 t) + x_n \sin((\omega_0 + \omega)t + \phi) \tag{5.14}$$

$$x(t) = x_0 \sin(\omega_0 t) + x_n \sin(\omega_0 t)\cos(\omega t + \phi) + x_n \cos(\omega_0 t)\sin(\omega t + \phi) \tag{5.15}$$

$$x(t) = \underbrace{[x_0 + x_n \cos(\omega t + \phi)]}_{A} \sin(\omega_0 t) + \underbrace{[x_n \sin(\omega t + \phi)]}_{B} \cos(\omega_0 t) \tag{5.16}$$

$$x(t) = A \sin(\omega_0 t) + B \cos(\omega_0 t) \tag{5.17}$$

The resultant time-varying amplitude at ω_0 is

$$R = \sqrt{A^2 + B^2} = \sqrt{x_0^2 + 2x_0 x_n \cos(\omega t + \phi) + x_n^2} \tag{5.18}$$

$$R = x_0 \sqrt{1 + \frac{2x_n}{x_0}\cos(\omega t + \phi) + \frac{x_n^2}{x_0^2}} \tag{5.19}$$

For a carrier drive much larger than the noise, i.e., $x_0 \gg x_n$:

$$R \approx x_0\,(1 + \frac{x_n}{x_0}\cos(\omega t + \phi)) = x_0\,(1 + \underbrace{\frac{x_n}{x_0}\sin\Big(\omega t + \phi + \frac{\pi}{2}\Big))}_{\substack{\text{Amplitude modulation} \\ \langle M^2\rangle = \frac{x_n^2}{2x_0^2}}} \tag{5.20}$$

The phase angle of $x(t)$ with respect to the pure carrier $\sin(\omega_0 t)$ is:

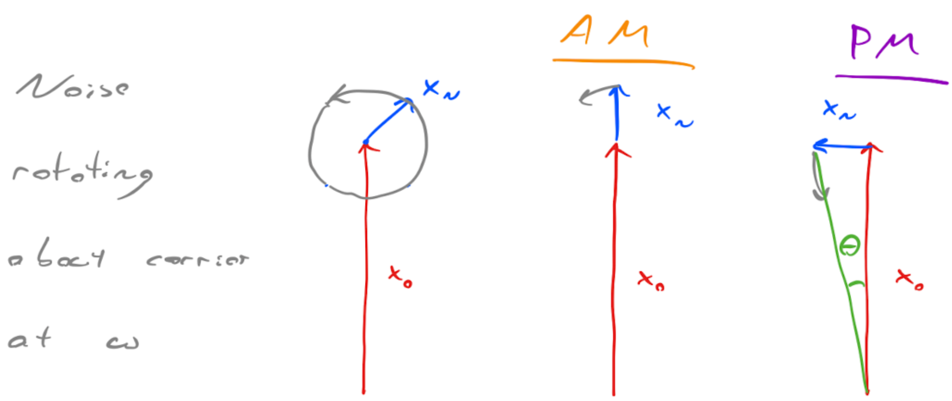
$$\tan(\Theta) = \frac{B}{A} = \frac{x_n \sin(\omega t + \phi)}{x_0 + x_n \cos(\omega t + \phi)} \tag{5.21}$$

For $x_0 \gg x_n$:

$$\tan(\Theta) \approx \frac{x_n}{x_0}\ \sin(\omega t + \phi) \tag{5.22}$$

$$\Theta \approx \underbrace{\frac{x_n}{x_0}\ \sin(\omega t + \phi)}_{\text{Phase modulation } \langle \Theta^2 \rangle = \frac{x_n^2}{2x_0^2}} \tag{5.23}$$

Phasor representation:



This means that noise in the sidebands of the carrier (at $\omega_0 + \omega$) can be represented as one half amplitude modulation noise and one half phase modulation noise. For frequency measurements or clocks, amplitude modulation is not important, only phase noise.

Averaging over time, we can relate the phase fluctuations at ω with the displacement noise at $\omega_0 + \omega$:

$$\langle \Theta^2 \rangle_\omega = \frac{x_n^2}{2x_0^2} = \frac{1}{x_0^2} \langle x^2 \rangle_{\omega_0+\omega} \tag{5.24}$$

Expressed in terms of the power spectral densities:

$$\frac{1}{2\pi} S_\Theta(\omega)\,\mathrm{d}\omega = \frac{1}{x_0^2}\left[\frac{1}{2\pi}\bar{S}_x(\omega_0+\omega)\,\mathrm{d}\omega\right] \tag{5.25}$$

$$\Longrightarrow \quad \boxed{S_\Theta(\omega) = \frac{1}{x_0^2}\,\bar{S}_x(\omega_0+\omega)} \quad \text{Double-sided} \tag{5.26}$$

$$\boxed{\bar{S}_\Theta(\omega) = \frac{2}{x_0^2}\,\bar{S}_x(\omega_0+\omega)} \quad \text{Single-sided} \tag{5.27}$$

If we now plug in our thermal PSD for $S_x(\omega)$, we can solve for the thermal phase PSD:

$$S_\Theta(\omega) = \frac{4k_B T\,\Gamma}{x_0^2 m^2}\,\frac{1}{(2\omega_0\omega + \omega^2)^2 + \frac{\Gamma^2(\omega_0^2+\omega^2)^2}{m^2}} \qquad \Gamma = \frac{m\omega_0}{Q} \tag{5.28}$$

For frequencies that are well off resonance, $\omega \gg \frac{\omega}{Q}$, but small compared to the resonance frequency, $\omega \ll \omega_0$:

$$S_\Theta(\omega) \approx \frac{k_B T\,\Gamma}{x_0^2 m^2}\,\frac{1}{\omega_0^2\omega^2} \tag{5.29}$$

$$S_\Theta(\omega) \approx \frac{k_B T\,\Gamma}{x_0^2 m^2 \omega_0^2}\,\frac{1}{\omega^2} = \frac{k_B T}{x_0^2 m\omega_0 Q}\,\frac{1}{\omega^2} \quad \text{Double-sided} \tag{5.30}$$

$$\bar{S}_\Theta(\omega) \approx \frac{2k_B T\,\Gamma}{x_0^2 m^2 \omega_0^2}\,\frac{1}{\omega^2} = \frac{2k_B T}{x_0^2 m\omega_0 Q}\,\frac{1}{\omega^2} \quad \text{Single-sided} \tag{5.31}$$

5.4 Dissipation-induced Frequency Noise

Frequency noise and phase noise are closely related. For a phase modulation Θ at frequency ω, we have a frequency modulation $\delta\omega_0 = \omega\,\Theta$ ($\rightarrow \delta\nu_0 = \frac{\omega}{2\pi}\,\Theta$).

$$\Longrightarrow S_\nu(\omega) = \left(\frac{\partial \delta\nu_0}{\partial \Theta}\right)^2 S_\Theta(\omega) \tag{5.32}$$

$$S_\nu(\omega) = \frac{\omega^2}{4\pi^2} S_\Theta(\omega) \tag{5.33}$$

$$S_\nu(\omega) = \frac{k_B T\,\Gamma}{\pi^2 x_0^2 m^2}\,\frac{\omega^2}{(2\omega_0\omega + \omega^2)^2 + \frac{\Gamma^2(\omega_0+\omega)^2}{m^2}} \tag{5.34}$$

Again, in the limit $\omega \gg \frac{\omega_0}{Q}$ and $\omega \ll \omega_0$:

$$S_\nu(\omega) \approx \frac{k_B T\,\Gamma}{4\pi^2 x_0^2 m^2 \omega_0^2} = \frac{k_B T}{4\pi^2 x_0^2 m \omega_0 Q} \qquad \text{Double-sided} \tag{5.35}$$

$$\bar{S}_\nu(\omega) \approx \frac{k_B T\,\Gamma}{2\pi^2 x_0^2 m^2 \omega_0^2} = \frac{k_B T}{2\pi^2 x_0^2 m \omega_0 Q} \qquad \text{Single-sided} \tag{5.36}$$

We see that within this approximation, the noise is white again, i.e., has no frequency dependence.

5.5 Measuring using a Harmonic Oscillator

Take a harmonic mechanical mode obeying the usual equation of motion:

$$m\ddot{x} + \Gamma\dot{x} + kx = 0\ , \tag{5.37}$$

where no forces are applied. The solution to this differential equation has the form:

$$x(t) = A\mathrm{e}^{i\omega t} \tag{5.38}$$

Therefore:

$$-\,m\omega^2 + i\Gamma\omega + k = 0 \tag{5.39}$$

$$\omega = \frac{i\frac{\Gamma}{m} \pm \sqrt{\frac{4k}{m} - \frac{\Gamma^2}{m^2}}}{2} \tag{5.40}$$

$$\omega = i\frac{\Gamma}{2m} \pm \sqrt{\frac{k}{m} + \frac{\Gamma^2}{4m^2}} \tag{5.41}$$

These solutions correspond to decaying harmonic oscillations. If we define $\omega_0 = \sqrt{k/m}$ and $\Gamma = m\omega_0/Q$,

$$\omega = i\frac{\omega_0}{2Q} \pm \omega_0 \underbrace{\sqrt{1 + \frac{1}{4Q^2}}}_{\text{This term} \approx 1 \text{ for } Q \gg 1\text{, i.e., small } \Gamma} \tag{5.42}$$

Therefore, for high Q, the resonant frequency is given by

$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}} \tag{5.43}$$

Plugging the result of equation (5.42) within this approximation into equation (5.38) yields:

$$x(t) = \mathrm{e}^{-\frac{\omega_0}{2Q}t}\left(A\mathrm{e}^{i\omega_0 t} + B\mathrm{e}^{-i\omega_0 t}\right) \tag{5.44}$$

The resonator's resonance frequency can be used to measure changes in mass, spring constant, and force gradients.

Measuring Mass

Let us take a small change in mass δm:

$$\omega_0 + \delta\omega_0 = \sqrt{\frac{k}{m + \delta m}} \tag{5.45}$$

$$\omega_0 + \delta\omega_0 = \omega_0 \sqrt{\frac{m}{m + \delta m}} \tag{5.46}$$

$$\omega_0 + \delta\omega_0 = \omega_0 \left(1 + \frac{\delta m}{m}\right)^{-1/2} \tag{5.47}$$

Expanding around small $\delta m/m$ and keeping only first order terms:

$$\omega_0 + \delta\omega_0 \approx \omega_0 \left(1 - \frac{1}{2}\frac{\delta m}{m}\right) \tag{5.48}$$

$$\frac{\delta\omega_0}{\omega_0} = -\frac{1}{2}\frac{\delta m}{m} \tag{5.49}$$

$$\boxed{\frac{\delta\nu_0}{\nu_0} = -\frac{1}{2}\frac{\delta m}{m}} \tag{5.50}$$

We see that the relative frequency shift is proportional to the relative mass change.

Since we now know the relationship between changes in resonance frequency and changes in mass, we can now write down a thermal limit for a minimum detectable mass, based on thermal frequency fluctuations $S_v(\omega)$. From above:

$$\delta m = -\frac{2m}{\nu_0}\,\delta\nu_0 = -\frac{4\pi m}{\omega_0}\,\delta\nu_0 \tag{5.51}$$

$$\bar{S}_m(\omega) = \left(-\frac{4\pi m}{\omega_0}\right)^2 \bar{S}_\nu(\omega) \tag{5.52}$$

$$\bar{S}_m(\omega) = \frac{16\pi^2 m^2}{\omega_0^2}\,\bar{S}_\nu(\omega) \tag{5.53}$$

Again, in the limit $\omega \gg \omega_0/Q$ and $\omega \ll \omega_0$:

$$\boxed{\bar{S}_m(\omega) = \frac{2}{x_0^2\omega_0^4}\,\bar{S}_f} \quad (5.54) \qquad \text{PSD of thermal "mass" fluctuations}$$

As a result, the minimum detectable mass due to thermal fluctuations is:

$$m_{\min} = \sqrt{\frac{8k_B T\,\Gamma\Delta\nu}{x_0^2\omega_0^4}} = \frac{2}{x_0\omega_0^2}\sqrt{2k_B T\,\Gamma\Delta\nu} \qquad [kg] \tag{5.55}$$

We see that, to improve sensitivity (i.e., make m_{min} smaller), we have to reduce T, Γ, $\Delta\nu$ and increase x_0 and ω_0 $\Rightarrow$ Cold, high frequency, low-loss resonators are best.

Measuring Force Gradients

Similarly, the system can be exposed to an external force gradient, $\frac{\partial F}{\partial x}$, or equivalently a change in spring constant δk. In each case, the equation of motion becomes:

$$m\ddot{x} + \Gamma\dot{x} + kx = \frac{\partial F}{\partial x}x \tag{5.56}$$

or

$$m\ddot{x} + \Gamma\dot{x} + (k + \delta k)x = 0 \tag{5.57}$$

In both cases, the analysis is the same, so we will proceed with δk ($\delta k = -\frac{\partial F}{\partial x}$).

We have:

$$\omega_0 + \delta\omega_0 = \sqrt{\frac{k + \delta k}{m}} \tag{5.58}$$

$$\omega_0 + \delta\omega_0 = \omega_0 \left(1 + \frac{\delta k}{k}\right)^{1/2} \tag{5.59}$$

For small $\delta k/k$:

$$\omega_0 + \delta\omega_0 \approx \omega_0 \left(1 + \frac{1}{2}\frac{\delta k}{k}\right) \tag{5.60}$$

$$\boxed{\frac{\delta\omega_0}{\omega_0} \approx \frac{1}{2}\frac{\delta k}{k} \qquad \frac{\delta\omega_0}{\omega_0} \approx -\frac{1}{2}\frac{\frac{\partial F}{\partial x}}{k}} \tag{5.61}$$

We see, the relative frequency shift is also proportional to force gradients and changes in stiffness.

Now that we have the relationship between changes in k (or external gradients) and frequency, we can state the thermo-mechanical limits imposed by frequency noise $S_\nu(\omega)$.

$$\delta k = \frac{2k}{\nu_0}\,\delta\nu_0 = \frac{4\pi k}{\omega_0}\delta\nu_0 \tag{5.62}$$

$$\delta k = 4\pi m\omega_0\delta\nu_0 \tag{5.63}$$

$$\bar{S}_k(\omega) = 16\pi^2 m^2\omega_0^2\bar{S}_\nu(\omega) \tag{5.64}$$

In the limit $\omega \gg \omega_0/Q$ and $\omega \ll \omega_0$:

$$\bar{S}_k(\omega) = 16\pi^2 m^2\omega_0^2\,\frac{k_B T\,\Gamma}{2\pi^2 x_0^2 m^2\omega_0^2} = \frac{8k_B T\,\Gamma}{x_0^2} \tag{5.65}$$

$$\boxed{S_k(\omega) = \frac{2}{x_0^2}\bar{S}_f} \tag{5.66}$$

PSD of thermal "spring constant"
or force gradient fluctuations

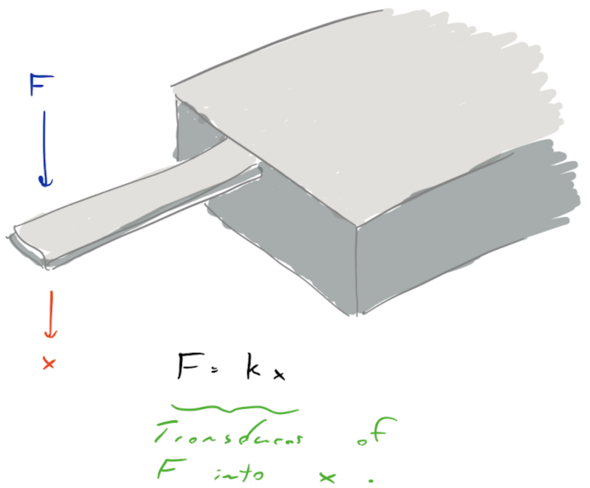
As a result, the minimum detectable spring constant change or force gradient due to thermal fluctuations is:

$$k_{\min} = \frac{2}{x_0}\sqrt{2k_B T\,\Gamma\Delta\nu} \qquad \left[\frac{N}{m}\right] \tag{5.67}$$

To improve sensitivity, the same measures should be taken as for optimizing $F_{\min}$: decrease T, Γ, $\Delta\nu$. In addition, the driving amplitude x_0 should be increased as much as possible without affecting spatial resolution and without entering a non-linear regime.

5.6 Displacement Measurement

I now want to discuss measurements of nanomechanical displacements. Nanomechanical elements are typically transducers, i.e., devices that convert force, torque, mass change, etc. into a displacement or phase shift. This displacement or phase shift then has to be measured by a detector. Such detectors of nanomechanical motion are typically optical or electronic. Together the transducer and the detector make the force, torque, or mass change sensor.



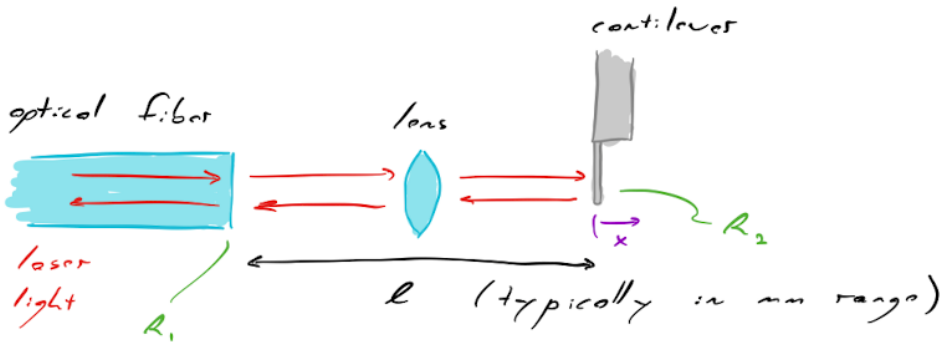
Now, displacement x must be detected by:

- tunneling
- optical deflection
- optical interferometry
- microwave interferometry
- magnetomotive effect
- piezoelectric effect
- capacitive effect

Many techniques have been developed and are used nowadays. See the lecture slides for examples.

5.7 Fiber Interferometry

We will now examine in detail displacement detection by optical interferometry.



Assume that the reflectivity of the fiber face is R_1 and of the cantilever is R_2. Assume low reflectivities, such that $R_1, R_2 \ll 1$. The incident laser power is $P_I = E_I^2$, where E_I is the electric field magnitude of the incident laser beam. The power reflected back down the fiber P_R depends on the interference between light reflected at R_1 and R_2:

$$P_R = \left| E_I \sqrt{R_1}\, \mathrm{e}^{i\phi_1} + E_I \sqrt{1 - R_1} \sqrt{R_2} \sqrt{1 - R_1}\, \mathrm{e}^{i\phi_2} \right|^2 \tag{5.68}$$

ϕ_1 and ϕ_2 are the phase at R_1 and R_2, respectively.

We can rearrange this:

$$P_R = E_I^2 \left[R_1 + (1 - R_1)^2 R_2 + 2\sqrt{R_1 R_2}(1 - R_1) \cos(\phi_1 - \phi_2) \right] \tag{5.69}$$

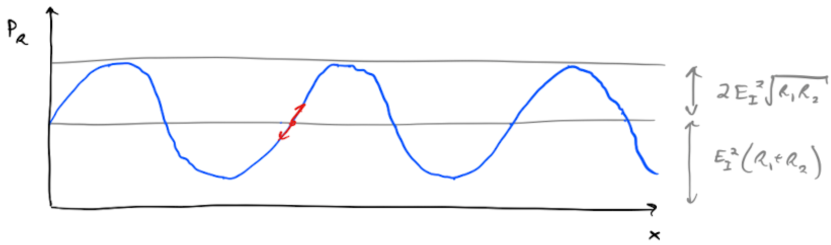
$$P_R = E_I^2 \left[R_1 + (1 - R_1)^2 R_2 + 2\sqrt{R_1 R_2}(1 - R_1) \cos\left(\frac{4\pi l}{\lambda}\right) \right] \tag{5.70}$$

If $l = x_0 + x$, with $x_0 = \lambda(n + 3/8)$, for $n = 1, 2, 3, ...$, then:

$$P_R = E_I^2 \left[R_1 + (1 - R_1)^2 R_2 + 2\sqrt{R_1 R_2}(1 - R_1) \sin\left(\frac{4\pi x}{\lambda}\right) \right] \tag{5.71}$$

Now, let us ignore higher orders of R_1 and R_2:

$$P_R \approx E_I^2 \left[R_1 + R_2 + 2\sqrt{R_1 R_2} \sin\left(\frac{4\pi x}{\lambda}\right) \right] \tag{5.72}$$



$$\text{Visibility:} \quad \frac{\text{Amplitude}}{\text{Average}} = \underbrace{\frac{2\sqrt{R_1 R_2}}{R_1 + R_2}}_{\text{Maximum for } R_1 = R_2 \text{ and } R_1, R_2 \to 1} \tag{5.73}$$

For $x \ll \lambda$:

$$\boxed{P_R \approx E_I^2 \left[R_1 + R_2 + \frac{8\pi}{\lambda}\sqrt{R_1 R_2}\, x \right]} \tag{5.74}$$

For small displacements, this is a proportional detector with efficiency

$$\epsilon = \frac{8\pi P_I \sqrt{R_1 R_2}}{\lambda} \qquad \left[\frac{W}{m}\right] \tag{5.75}$$

This optical power in Watts is then transformed into an electric current by a photodiode with efficiency S, such that $I_{\text{PD}} = SP_R$. The main sources of noise will be electronic noise I_e, shot noise I_{shot}, and mechanical vibration noise I_m.

$$I_e \propto \text{constant} \tag{5.76}$$

$$I_m = S\,\epsilon\, x_{\text{noise}} \propto P_I \qquad x_{\text{noise}}\text{: mechanical vibrations} \tag{5.77}$$

We see that the current noise I_e is independent of optical power, but the mechanical noise I_m is proportional with incident power.

Aside: Shot Noise

If the electron arrival events N making up a current are uncorrelated, then they are governed by a Poisson distribution (mean = variance). We have:

- Average current: $\langle I \rangle = K\, e$
- Average number of electron arrivals in τ: $\langle n \rangle = K\, \tau$
- Variance: $\langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle$ (Poissonian)

Therefore:

$$\langle I \rangle = \frac{e}{\tau} \langle n \rangle \tag{5.78}$$

$$\langle I^2 \rangle = \left\langle \left(\frac{n\, e}{\tau} \right)^2 \right\rangle = \frac{e^2}{\tau^2} \langle n^2 \rangle \tag{5.79}$$

$$\langle \Delta I^2 \rangle = \langle I^2 \rangle - \langle I \rangle^2 = \frac{e^2}{\tau^2} \left(\langle n^2 \rangle - \langle n \rangle^2 \right) \tag{5.80}$$

$$\langle \Delta I^2 \rangle = \frac{e^2}{\tau^2} \langle n \rangle = \frac{e}{\tau} \langle I \rangle \tag{5.81}$$

$$\Delta I = \sqrt{\frac{e}{\tau} \langle I \rangle} \tag{5.82}$$

If we consider a bandwidth $\Delta f = 1/2\tau$ for an averaging filter and $I = \langle I \rangle$, then

$$\boxed{I_{\text{shot}} = \sqrt{2e\, I \Delta f}} \tag{5.83}$$

So the current shot noise in our photodetector will depend on the photodiode current I_{PD}:

$$I_{\text{shot}} = \sqrt{2e\, I_{\text{PD}} \Delta f} = \sqrt{2e\, S P_R \Delta f} \tag{5.84}$$

$$I_{\text{shot}} \propto \sqrt{P_R} \propto \sqrt{P_I} \tag{5.85}$$

The signal-to-noise ratio (SNR) of our displacement detection will depend on what kind of noise dominates.

<u>I_e dominates:</u>

$$\text{SNR} = \frac{I_{\text{sig}}}{I_{\text{noise}}} = \frac{S\, \epsilon\, x_{\text{sig}}}{I_e} \tag{5.86}$$

$$\text{SNR} = \frac{8\pi}{\lambda} \sqrt{R_1 R_2}\, \frac{S\, P_I}{I_e}\, x_{\text{sig}} \tag{5.87}$$

$$\text{SNR} \propto P_I \qquad \rightarrow \text{ linear with laser power} \tag{5.88}$$

<u>I_m dominates:</u>

$$\text{SNR} = \frac{I_{\text{sig}}}{I_{\text{noise}}} = \frac{S\, \epsilon\, x_{\text{sig}}}{S\, \epsilon\, x_{\text{noise}}} \tag{5.89}$$

$$\text{SNR} = \frac{x_{\text{sig}}}{x_{\text{noise}}} \qquad \rightarrow \text{ independent of laser power} \tag{5.90}$$

$\underline{I_{\text{shot}}\text{ dominates:}}$

$$\text{SNR} = \frac{I_{\text{sig}}}{I_{\text{noise}}} = \frac{S\,\epsilon\,x_{\text{sig}}}{\sqrt{2e\,SP_R\Delta f}} \tag{5.91}$$

$$\text{SNR} \propto \frac{P_I}{\sqrt{P_I}} \propto \sqrt{P_I} \qquad \rightarrow \text{ dependent on square root of laser power} \tag{5.92}$$

Therefore, the type of limiting noise can be determined by the dependence of SNR on the laser power P_I. Ideally, shot noise dominates the noise, as it is a fundamentally unavoidable source of noise. In shot noise limit, the ultimate noise floor of the detector is given by:

$$I_{\text{sig}} = I_{\text{shot}} \tag{5.93}$$

$$S\,\epsilon\,x_{\text{shot}} = \sqrt{2e\,SP_R\Delta f} \tag{5.94}$$

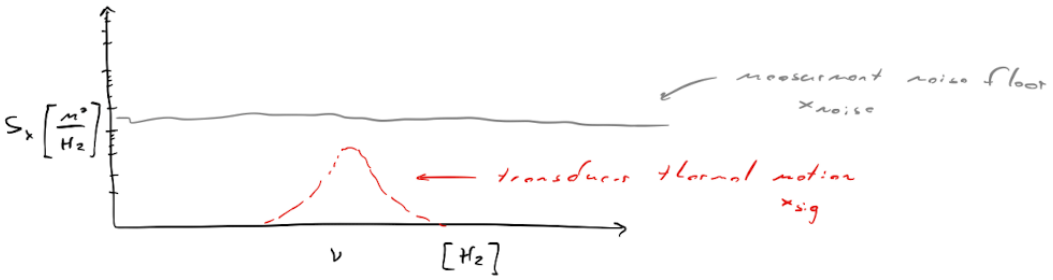
$$S\,\frac{8\pi}{\lambda}\,P_I\sqrt{R_1R_2}\,x_{\text{shot}} = \sqrt{2e\,S\,P_I(R_1+R_2)\Delta f} \tag{5.95}$$

Therefore, the shot-noise floor for interferometric detection is given by:

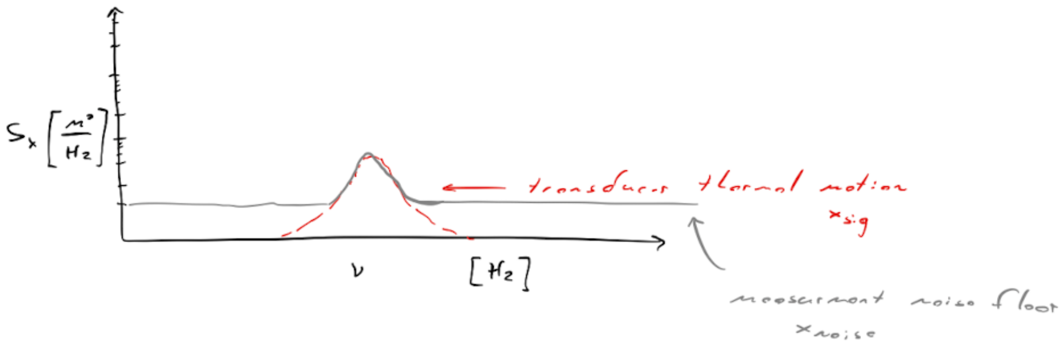
$$\boxed{x_{\text{shot}} = \frac{\lambda}{8\pi}\sqrt{\frac{2e\Delta f}{SP_I}}\sqrt{\frac{R_1+R_2}{R_1R_2}}} \tag{5.96}$$

with signal to noise ratio (in the shot noise limit)

$$\text{SNR} = \frac{x_{\text{sig}}}{x_{\text{noise}}} \propto \sqrt{P_I} \tag{5.97}$$

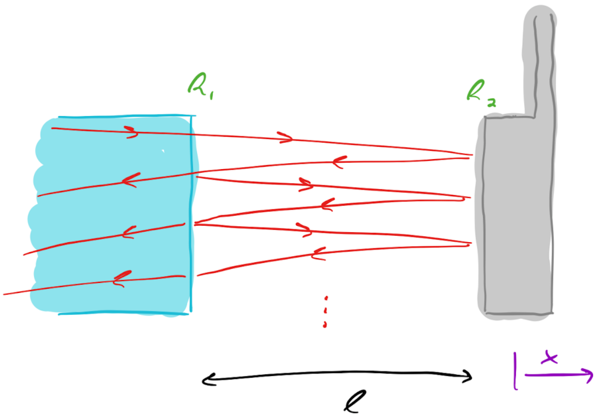


Increasing P_I leads then to:



5.8 Interferometric Detection in Fabry-Perot Limit

Consider large R_1 and R_2, where multiple reflections cannot be ignored:



Again, the laser power is given by the amplitude square of the electric field:

$$P_I = |E_I|^2 \qquad P_R = |E_R|^2 \qquad P_T = |E_T|^2 \tag{5.98}$$

Focusing on the transmitted electric field:

$$E_T = E_I \left[\sqrt{T_1}\sqrt{T_2}\, e^{i\frac{2\pi l}{\lambda}} + \sqrt{T_1}\sqrt{R_2}\sqrt{R_1}\sqrt{T_2}\, e^{i\frac{6\pi l}{\lambda}} + ... \right] \tag{5.99}$$

$$E_T = E_I \sqrt{T_1 T_2}\, e^{i\frac{2\pi l}{\lambda}} \underbrace{\left[\sum_{n=0}^{\infty} \left(\sqrt{R_1 R_2}\, e^{i\frac{4\pi l}{\lambda}} \right)^n \right]}_{\frac{1}{1-\sqrt{R_1 R_2}\, e^{i\frac{4\pi l}{\lambda}}}} \tag{5.100}$$

$$E_T = E_I \frac{\sqrt{T_1 T_2}\, e^{i\frac{2\pi l}{\lambda}}}{1 - \sqrt{R_1 R_2}\, e^{i\frac{4\pi l}{\lambda}}} \tag{5.101}$$

This results in a transmitted laser power:

$$P_T = |E_T|^2 = E_I^2 \frac{T_1 T_2}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}\underbrace{\cos\left(\frac{4\pi l}{\lambda}\right)}_{1-2\sin^2\left(\frac{2\pi l}{\lambda}\right)}} \tag{5.102}$$

$$P_T = |E_T|^2 = P_I \frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1 R_2})^2 + 4\sqrt{R_1 R_2}\sin^2\left(\frac{2\pi l}{\lambda}\right)} \tag{5.103}$$

$$P_T = P_I \frac{\frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1 R_2})^2}}{1 + \underbrace{\frac{4\sqrt{R_1 R_2}}{(1-\sqrt{R_1 R_2})^2}}_{\text{Finesse } F} \sin^2\left(\frac{2\pi l}{\lambda}\right)} \tag{5.104}$$

$$P_R = P_I - P_T \tag{5.105}$$

Finally, we arrive at the expression for the reflected power:

$$\boxed{P_R = P_I \left[1 - \frac{\frac{(1-R_1)(1-R_2)}{(1-\sqrt{R_1 R_2})^2}}{1 + F \sin^2\left(\frac{2\pi l}{\lambda}\right)} \right]} \tag{5.106}$$

This expression depends sensitively on cavity length l !

For small x around x_0:

$$P_R = P_R(x_0) + \frac{\partial P_R}{\partial l}(x_0) \cdot x \tag{5.107}$$

$$\implies I_{\text{sig}} = S \frac{\partial P_R}{\partial l}(x_0) \cdot x_{\text{sig}} \tag{5.108}$$

In the shot noise limit:

$$I_{\text{shot}} \approx \sqrt{2e\, S P_R(x_0) \Delta f} \tag{5.109}$$

$$I_{\text{sig}} = I_{\text{shot}} \tag{5.110}$$

$$S \frac{\partial P_R}{\partial l}(x_0) \cdot x_{\text{shot}} = \sqrt{2e\, S P_R(x_0) \Delta f} \tag{5.111}$$

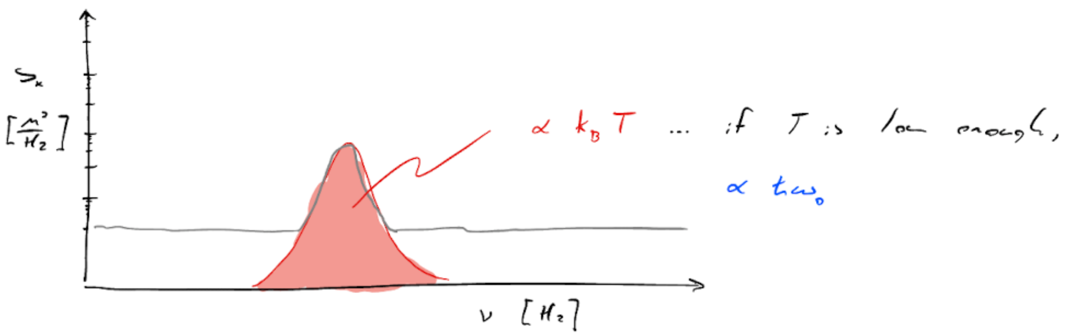
$$\boxed{x_{\text{shot}} = \sqrt{\frac{2e\, P_R(x_0) \Delta f}{S}} \; \frac{1}{\frac{\partial P_R}{\partial l}(x_0)}} \tag{5.112}$$

6 Cooling Mechanical Resonators - Basics

With mechanical resonators, cooled to ultra-low temperatures, the ultimate force resolution can be achieved. Also, one can reach the quantum regime, with interesting properties as mechanical superpositions and coherences. To this end, the following relation between resonance frequency ω and temperature T should be met:

$$\hbar\omega \gg k_B T \tag{6.1}$$

Then, the system is in its quantum mechanical ground state. What does that mean?



Let us first think of our mechanical mode of interest in our resonator as a harmonic oscillator, as usual. We can then calculate its zero-point energy using principles of statistical mechanics and quantum mechanics.

6.1 Zero-point Energy: Quantum Harmonic Oscillator

Recall the energies of quantum harmonic oscillators:

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right) \qquad \text{for } n = 0, 1, 2, ... \tag{6.2}$$

$$E_0 = \frac{\hbar\omega_0}{2} \qquad \longleftarrow \text{zero-point energy} \tag{6.3}$$

The expectation value of x^2 is given by:

$$\langle x^2 \rangle = \langle n|\hat{x}^2|n\rangle = \frac{\hbar}{m\omega_0}\left(n + \frac{1}{2}\right) \tag{6.4}$$

For $n = 0$, i.e., in the ground state:

$$\langle x^2 \rangle_0 = \frac{\hbar}{2m\omega_0} = x_{\text{ZPF}}^2 \qquad \longleftarrow \text{Zero-point fluctuations} \tag{6.5}$$

From here, we can obtain the *Standard Quantum Limit*:

$$\boxed{\Delta x_{\text{SQL}} = x_{\text{ZPF}} = \sqrt{\langle x^2 \rangle_0}} \tag{6.6}$$

with the wave function for $n = 0$ of the harmonic oscillator:

$$\boxed{\Psi_0(x) = \left(\frac{1}{2\pi x_{\text{ZPF}}^2}\right)^{1/4} \exp\left[-\left(\frac{x}{2x_{\text{ZPF}}}\right)^2\right]} \tag{6.7}$$

According to statistical mechanics, the mean energy $\bar{E}$ in a quantum harmonic oscillator at temperature T is given by:

$$\bar{E} = \frac{\sum_{\mathrm{n=0}}^{\infty} \mathrm{e}^{-\beta E_n}\ E_n}{\sum_{\mathrm{n=0}}^{\infty} \mathrm{e}^{-\beta E_n}} \quad \text{with} \beta = \frac{1}{k_B T} \tag{6.8}$$

$$\bar{E} = -\frac{1}{\mathscr{Z}}\frac{\partial \mathscr{Z}}{\partial \beta} = -\frac{\partial \ln \mathscr{Z}}{\partial \beta}\ , \tag{6.9}$$

where we used the partition function

$$\mathscr{Z} = \sum_{\mathrm{n=0}}^{\infty} \mathrm{e}^{-\beta E_n} \tag{6.10}$$

$$= \mathrm{e}^{-\frac{1}{2}\beta\hbar\omega_0} \sum_{\mathrm{n=0}}^{\infty} \mathrm{e}^{-n\beta\hbar\omega_0} \tag{6.11}$$

$$= \mathrm{e}^{-\frac{1}{2}\beta\hbar\omega_0} \left(1 + \mathrm{e}^{-\beta\hbar\omega_0} + \mathrm{e}^{-2\beta\hbar\omega_0} + ...\right) \tag{6.12}$$

$$= \mathrm{e}^{-\frac{1}{2}\beta\hbar\omega_0} \left(\frac{1}{1-\mathrm{e}^{-\beta\hbar\omega_0}}\right) \tag{6.13}$$

Therefore:

$$\ln \mathscr{Z} = -\frac{1}{2}\beta\hbar\omega_0 - \ln\left(1 - \mathrm{e}^{-\beta\hbar\omega_0}\right) \tag{6.14}$$

Plugging this result into equation 6.9 yields

$$\bar{E} = -\frac{\partial \ln \mathscr{Z}}{\partial \beta} = -\left(-\frac{1}{2}\hbar\omega_0 - \frac{\mathrm{e}^{-\beta\hbar\omega_0}\hbar\omega_0}{1-\mathrm{e}^{-\beta\hbar\omega_0}}\right) \tag{6.15}$$

$$\boxed{\bar{E} = \hbar\omega_0\left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\hbar\omega_0/k_B T} - 1}\right)} \tag{6.16}$$

For $k_B T \gg \hbar\omega_0$:

$$\bar{E} \approx \hbar\omega_0\left(\frac{1}{2} + \frac{k_B T}{\hbar\omega_0}\right) \tag{6.17}$$

$$\boxed{\bar{E} \approx k_B T} \quad \longleftarrow \text{Equipartition} \tag{6.18}$$

For $k_B T \ll \hbar\omega_0$:

$$\boxed{\bar{E} \approx \hbar\omega_0\left(\frac{1}{2} + \mathrm{e}^{-\hbar\omega_0/k_B T}\right)} \quad \longleftarrow \lim_{T\to 0} \bar{E} = E_0 \tag{6.19}$$

6.2 Zero-point motion: Quantum Harmonic Oscillator

For the harmonic oscillator, we have:

$$\langle x^2 \rangle = \langle n|\hat{x}^2|n\rangle = \frac{\hbar}{m\omega_0}\left(n + \frac{1}{2}\right) = \frac{E_n}{m\omega_0^2} \tag{6.20}$$

$$\overline{\langle x^2 \rangle} = \frac{\bar{E}}{m\omega_0^2} \tag{6.21}$$

$$\boxed{\overline{\langle x^2 \rangle} = x_{\text{ZPF}}^2 + 2x_{\text{ZPF}}^2 \left(\frac{1}{e^{\hbar\omega/k_B T} - 1} \right)} \tag{6.22}$$

So how do we put nanomechanical oscillators into a quantum state, specifically, the ground state? One way is by brute force: picking a high-frequency mode and putting it in a cold fridge, such that:

$$\hbar\omega_0 \gg k_B T \tag{6.23}$$

This comes with some technical challenges, as the resonator fabrication (high frequency, low dissipation, low mass), then the displacement sensing (very low measurement imprecision, i.e., low noise floor), and also the refrigeration (reaching and maintaining mK temperatures).

Alternatively, one can try to cool the mechanical mode of interest - rather than the entire bath - by a number of techniques. One of these is *feedback cooling*, which we will discuss in the next chapter.

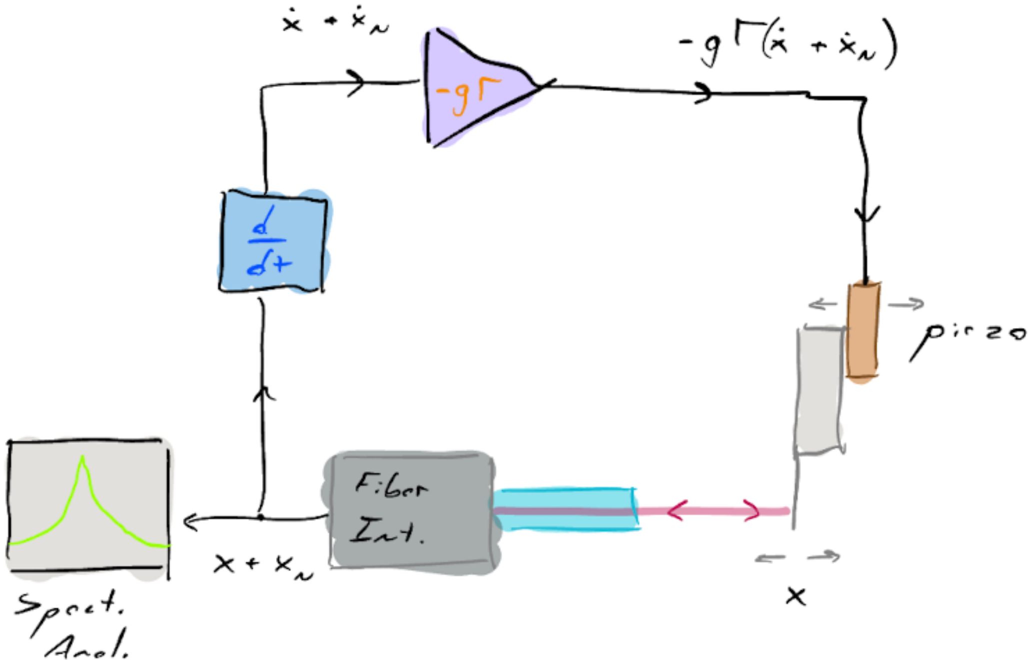
7 Feedback Cooling

Take the equation of motion for the harmonic oscillator as our model for e mechanical mode:

$$m\ddot{x} + \Gamma\dot{x} + kx = f(t) \quad \longleftarrow \text{thermal force} \tag{7.1}$$

$$\text{with} \quad \Gamma = \frac{m\omega_0}{Q} \ , \ k = m\omega_0^2 \tag{7.2}$$

We now implement a detection and feedback setup:



Feedback gives a modified equation of motion:

$$m\ddot{x} + \Gamma\dot{x} + kx = f(t) - g\Gamma(\dot{x} + \dot{x}_n) \ , \tag{7.3}$$

with x_n the measurement noise.

Let us look at one Fourier component:

$$-m\omega^2\hat{x}(\omega) + i\omega\Gamma\hat{x}(\omega) + k\hat{x}(\omega) = \hat{f}(\omega) - ig\Gamma\omega[\hat{x}(\omega) + \hat{x}_n(\omega)] \tag{7.4}$$

$$\implies \hat{x}(\omega) = \frac{\hat{f}(\omega) - ig\Gamma\omega\hat{x}_n(\omega)}{(k - m\omega^2) + i\omega\Gamma(1+g)} \tag{7.5}$$

Since $\left|\frac{a+ib}{c+id}\right|^2 = \frac{a^2+b^2}{c^2+d^2}$ and $S_x(\omega) = \lim_{T\to\infty} \frac{\hat{x}(\omega)\,\hat{x}^*(\omega)}{T}$:

$$\bar{S}_x(\omega) = \frac{\bar{S}_f + g^2\Gamma^2\omega^2\bar{S}_{x_n}(\omega)}{(k - m\omega^2)^2 + \omega^2\Gamma^2(1+g)^2} \tag{7.6}$$

The power spectral density of the mode's displacement can be expressed as:

$$\boxed{\begin{aligned}\bar{S}_x(\omega) = &\left[\frac{1}{(k-m\omega^2)^2+\omega^2\Gamma^2(1+g)^2}\right]\bar{S}_f(\omega)\\ &+\left[\frac{g^2\Gamma^2\omega^2}{(k-m\omega^2)^2+\omega^2\Gamma^2(1+g)^2}\right]\bar{S}_{x_n}(\omega)\end{aligned}} \tag{7.7}$$

What do we measure?

$$\hat{x}_{\text{meas}}(\omega) = \hat{x}(\omega) + \hat{x}_n(\omega) \tag{7.8}$$

$$\hat{x}_{\text{meas}}(\omega) = \frac{\hat{f}(\omega) - ig\Gamma\omega\hat{x}_n(\omega)}{(k-m\omega^2)+i\omega\Gamma(1+g)} + \hat{x}_n(\omega) \tag{7.9}$$

$$\hat{x}_{\text{meas}}(\omega) = \frac{\hat{f}(\omega) + \left[(k-m\omega^2)+i\omega\Gamma\right]\hat{x}_n(\omega)}{(k-m\omega^2)+i\omega\Gamma(1+g)} \tag{7.10}$$

Assume the thermal force noise $\hat{f}(\omega)$ and the measured displacement noise $\hat{x}_n(\omega)$ are uncorrelated. Therefore, the noises add in quadrature, yielding the power spectral density of the measured signal:

$$\bar{S}_{\text{meas}}(\omega) = \frac{\bar{S}_{\text{f}}(\omega) + \left[(k-m\omega^2)+\omega^2\Gamma^2\right]\bar{S}_{x_n}(\omega)}{(k-m\omega^2)^2+\omega^2\Gamma^2(1+g)^2} \tag{7.11}$$

$$\boxed{\begin{aligned}\bar{S}_{\text{meas}}(\omega) = &\left[\frac{1}{(k-m\omega^2)^2+\omega^2\Gamma^2(1+g)^2}\right]\bar{S}_f(\omega)\\ &+\left[\frac{(k-m\omega^2)^2+\omega^2\Gamma^2}{(k-m\omega^2)^2+\omega^2\Gamma^2(1+g)^2}\right]\bar{S}_{x_n}(\omega)\end{aligned}} \tag{7.12}$$

$$\text{Recall from earlier:}\quad \bar{S}_f(\omega) = 4k_BT\Gamma \tag{7.13}$$

$$\bar{S}_{x_n}(\omega) = \text{constant} \quad \longleftarrow \text{ typically from shot noise} \tag{7.14}$$

By damping with gain g we reduce the fluctuations in the mode of interest. This means that we cool the mode. Recall:

$$\langle x^2\rangle = \frac{1}{2\pi}\int_0^\infty \bar{S}_x(\omega)\,\mathrm{d}\omega \tag{7.15}$$

Equipartition says:

$$\frac{1}{2}k_BT_{\text{mode}} = \frac{1}{2}k\langle x^2\rangle \tag{7.16}$$

$$T_{\text{mode}} = \frac{k}{k_B}\langle x^2\rangle \tag{7.17}$$

$$\implies T_{\text{mode}} = \frac{k}{2\pi k_B}\int_0^\infty \bar{S}_x(\omega)\,\mathrm{d}\omega \tag{7.18}$$

Inserting (7.12) into (7.18) and integrating yields:

$$T_{\text{mode}} = \frac{T}{1+g} + \frac{k\Gamma}{4k_Bm}\left(\frac{g^2}{1+g}\right)\bar{S}_{x_n} \tag{7.19}$$

Minimizing this expression with respect to g results in

$$T_{\text{mode,min}}\sqrt{\frac{k\Gamma T}{k_B m}\bar{S}_{x_n}(\omega)} = \omega_0\sqrt{\frac{\Gamma T}{k_B}\bar{S}_{x_n}} \tag{7.20}$$

This is equivalent to a minimum phonon number:

$$N_{\text{mode, min}} = \frac{k_B T_{\text{mode, min}}}{\hbar\omega_0} = \frac{1}{\hbar}\sqrt{\underbrace{\Gamma k_B T}_{1/4\,\bar{S}_f}\bar{S}_{x_n}} \tag{7.21}$$

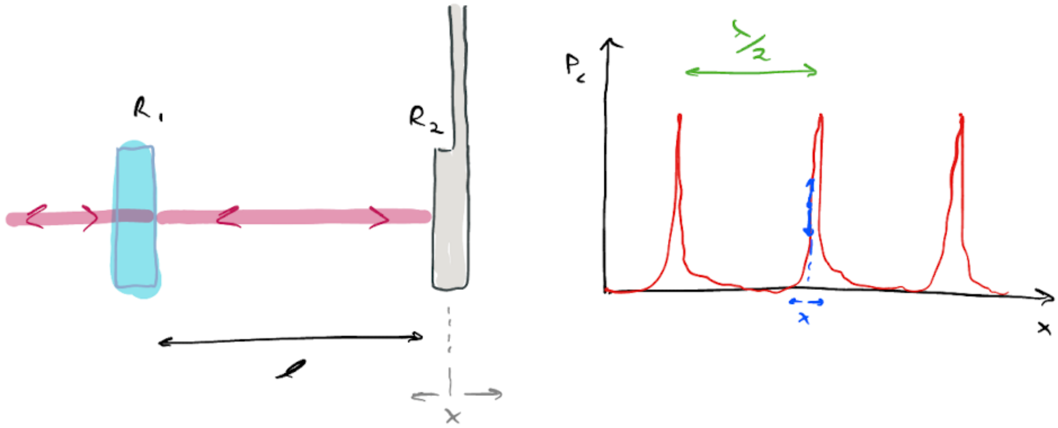
$$\boxed{N_{\text{mode, min}} = \frac{1}{2\hbar}\sqrt{\bar{S}_f\bar{S}_{x_n}} = \frac{1}{\hbar}\sqrt{S_f S_{x_n}}} \tag{7.22}$$

From equation (7.21), we see that for minimal phonon numbers, we want low T, Γ, and $\bar{S}_{x_n}$. I.e., a cold fridge, good resonators, and a sensitive measurement.

The optimum in $N_{\text{mode, min}}$ is achieved, as will be discussed, when $\bar{S}_f$ is dominated by detector back-action and $\bar{S}_{x_n}$ is quantum limited. More on this to come.

8 Cavity Cooling

So far, we have encountered brute-force cooling, and feedback cooling. Another method is cooling of a mechanical mode by an optical cavity. Recall our Fabry-Perot cavity:



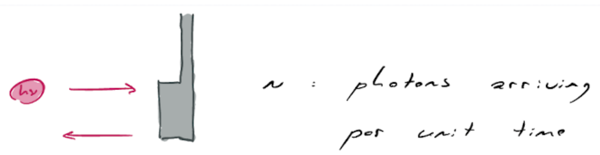
The power in the cavity is given by

$$P_C = P_I \left[\frac{\frac{1-R_1}{\left(1-\sqrt{R_1 R_2}\right)^2}}{1 + F\,\sin^2\left(\frac{2\pi l}{\lambda}\right)} \right] \quad \text{, where the Finesse } F = \frac{4\sqrt{R_1 R_2}}{\left(1-\sqrt{R_1 R_2}\right)^2} \tag{8.1}$$

Optical forces act on the compliant mirror. Most prominently, these are photo-thermal and radiation pressure. Both forces act with a characteristic time delay. We will now consider only the radiation pressure force.

8.1 Semi-classical Picture Cavity Cooling

As a first step, we consider a quantization of the photons, but treat the cantilever classically.



The momentum kick to the lever for each photon:

$$\Delta p = 2\frac{h\nu}{c} \tag{8.2}$$

$$\Delta p = \frac{2n\Delta t h\nu}{c} \quad \longleftarrow \text{momentum in } \Delta t \text{ with photon flux n} \tag{8.3}$$

$$\frac{\Delta p}{\Delta t} = \frac{2nh\nu}{c} \tag{8.4}$$

$$\text{Radiation force } \longrightarrow \quad \mathscr{F} = \frac{2P}{c} \quad \longleftarrow P: \text{ power (intensity)} \tag{8.5}$$

The force $\mathscr{F}$ is not instantaneous. The finesse F introduces a time lag in the radiation force. We can therefore write that the actual radiation force F_{opt} lags $\mathscr{F}$ a little bit:

$$\dot{F}_{\text{opt}}(t) = \frac{\mathscr{F}(x) - F_{\text{opt}}(t)}{\tau} \tag{8.6}$$

where τ is a time delay introduced by the optical cavity. Rearranging yields:

$$\tau \dot{F}_{\text{opt}}(t) + F_{\text{opt}}(t) = \mathscr{F}(x) \tag{8.7}$$

Let us focus on the Fourier component at ω:

$$i\omega\tau \hat{F}_{\text{opt}}(\omega) + \hat{F}_{\text{opt}}(\omega) = \hat{\mathscr{F}}(\omega) \tag{8.8}$$

$$\hat{F}_{\text{opt}}(\omega) = \frac{\hat{\mathscr{F}}(\omega)}{1 + i\omega\tau} \tag{8.9}$$

Linearizing with respect to small displacements x from the equilibrium position x_0 results in

$$\hat{F}_{\text{opt}}(\omega) = \frac{\mathscr{F}'(x_0)\,\hat{x}(\omega)}{1 + i\omega\tau} \tag{8.10}$$

If we now put this optical force into our equation of motion for the harmonic oscillator, we have:

$$m\ddot{x} + \Gamma\dot{x} + kx = f(t) + F_{\text{opt}} \tag{8.11}$$

For one Fourier component at ω:

$$-m\omega^2\hat{x}(\omega) + i\omega\Gamma\hat{x}(\omega) + k\hat{x}(\omega) = \hat{f}(\omega) + \underbrace{\frac{\mathscr{F}'(x_0)\,\hat{x}(\omega)}{1 + i\omega\tau}}_{\frac{\mathscr{F}'(x_0)\,\hat{x}(\omega)}{1+\omega^2\tau^2} - i\,\frac{\omega\tau\mathscr{F}'(x_0)\,\hat{x}(\omega)}{1+\omega^2\tau^2}} \tag{8.12}$$

$$-\,m\omega^2\hat{x}(\omega) + i\omega\underbrace{\left[\Gamma + \frac{\tau\mathscr{F}'(x_0)}{1+\omega^2\tau^2}\right]}_{\substack{\text{change in}\\ \text{damping (Q)}}}\hat{x}(\omega) + \underbrace{\left[k - \frac{\mathscr{F}'(x_0)}{1+\omega^2\tau^2}\right]}_{\substack{\text{change in spring}\\ \text{constant (freq.)}}}\hat{x}(\omega) = \hat{f}(\omega) \tag{8.13}$$

$$\text{Optical spring:} \qquad -\frac{\mathscr{F}'(x_0)}{1+\omega^2\tau^2} = k_{\text{opt}} \tag{8.14}$$

$$\text{Optical damping:} \qquad \frac{\tau'(x_0)}{1+\omega^2\tau^2} = \Gamma_{\text{opt}} \tag{8.15}$$

We can now go through the same analysis as with the feedback cooling. We simply use a renormalized spring constant k', which includes the optical spring effect and we use $g = \frac{1}{\Gamma}\frac{\tau\mathscr{F}'(x_0)}{1+\omega^2\tau^2}$. The analysis follows exactly the analysis for feedback cooling. In the limit where the measurement noise x_n is negligible:

$$T_{\text{mode}} \approx \frac{T}{1+g} \tag{8.16}$$

$$T_{\text{mode}} \approx T\,\frac{1}{1 + \frac{1}{\Gamma}\frac{\tau\mathscr{F}'(x_0)}{1+\omega^2\tau^2}} \tag{8.17}$$

$$\boxed{T_{\text{mode}} \approx T\left(\frac{\Gamma}{\Gamma + \Gamma_{\text{opt}}}\right)} \tag{8.18}$$

8.2 Quantum Picture - Power Spectral Densities

In order to discuss cavity cooling - and any other cooling for that matter - which reaches $k_B T \sim \hbar\omega_0$, we have to introduce a quantum mechanical treatment of the oscillator as well. First of all, we have to define a quantum analogue to the classical spectral density, i.e., the quantum spectral density. For example for quantum noise on the displacement of an oscillator, given by the quantum operator $\hat{x}$, we define:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \langle \hat{x}(t)\, \hat{x}(0) \rangle \, \mathrm{e}^{i\omega t} \, \mathrm{d}t \ , \tag{8.19}$$

where the double subscript indicates the quantum nature of the PSD.

$$\text{Compare to the classical analogue:} \quad S_x(\omega) = \int_{-\infty}^{\infty} \langle x(t)\, x(0) \rangle \, \mathrm{e}^{i\omega t} \, \mathrm{d}t \tag{8.20}$$

Here, $\hat{x}$ is a quantum operator and $\langle ... \rangle$ is a quantum statistical average using a density matrix.

Let us first consider a simple harmonic oscillator with no dissipation, with momentum operator $\hat{p}$, mass m, angular frequency ω_0, and position operator $\hat{x}$. The Hamiltonian for this system reads:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} \tag{8.21}$$

Using the Heisenberg picture we have for any operator $\hat{A}$:

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}\right] + \frac{\partial \hat{A}}{\partial t} \tag{8.22}$$

$$\Longrightarrow \quad \frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}, \hat{x}\right] + \frac{\partial \hat{x}}{\partial t} = \frac{\hat{p}}{m} \qquad \text{Recall: } [\hat{x}, \hat{p}] = i\hbar \tag{8.23}$$

$$\Longrightarrow \quad \frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}, \hat{p}\right] + \frac{\partial \hat{p}}{\partial t} = -m\omega_0^2 \hat{x} \tag{8.24}$$

Additional differentiation with respect to time yields

$$\frac{\mathrm{d}^2\hat{x}}{\mathrm{d}t^2} + \omega_0^2 \hat{x} = 0 \qquad\qquad \frac{\mathrm{d}^2\hat{p}}{\mathrm{d}t^2} + \omega_0^2 \hat{p} = 0 \tag{8.25}$$

Solution take the form:

$$\hat{x}(t) = \hat{x}(0)\cos(\omega_0 t) + \frac{\hat{p}(0)}{m}\sin(\omega_0 t) \tag{8.26}$$

$$\hat{p}(t) = \hat{p}(0)\cos(\omega_0 t) - m\omega_0 \hat{x}(0)\sin(\omega_0 t) \tag{8.27}$$

The correlation function thus looks like:

$$\left\langle \hat{x}(t)\, \hat{x}(0) \right\rangle = \left\langle \hat{x}(0)\, \hat{x}(0) \right\rangle \cos(\omega t) + \left\langle \hat{p}(0)\, \hat{x}(0) \right\rangle \frac{1}{m\omega_0} \sin(\omega_o t) \tag{8.28}$$

Let us now introduce the *creation* and *annihilation operators* $\hat{a}^+$ and $\hat{a}$:

$$\hat{x} = x_{\text{ZPF}}(\hat{a}^+ + \hat{a}) \qquad\qquad \hat{p} = \frac{i\hbar}{2x_{\text{ZPF}}}(\hat{a}^+ - \hat{a}) \ , \tag{8.29}$$

$$\text{where} \quad x_{\text{ZPF}}^2 = \langle 0|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega_0} \tag{8.30}$$

$$\text{and} \quad \left[\hat{a}, \hat{a}^+\right] = 1 \ , \quad \hat{n} = \hat{a}^+\hat{a} \tag{8.31}$$

In thermal equilibrium, we have:

$$\langle \hat{x}\,\hat{p}\rangle = \left\langle \frac{i\hbar}{2}\left(\hat{a}^+\hat{a}^+ - \hat{a}^+\hat{a} + \hat{a}\hat{a}^+ - \hat{a}\hat{a}\right) \right\rangle = \frac{i\hbar}{2} \tag{8.32}$$

$$\langle \hat{p}\,\hat{x}\rangle = \left\langle \frac{i\hbar}{2}\left(\hat{a}^+\hat{a}^+ + \hat{a}^+\hat{a} - \hat{a}\hat{a}^+ - \hat{a}\hat{a}\right) \right\rangle = -\frac{i\hbar}{2} \tag{8.33}$$

$$\langle \hat{x}\,\hat{x}\rangle = \left\langle x_{\text{ZPF}}^2\left(\hat{a}^+\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+ + \hat{a}\hat{a}\right) \right\rangle \tag{8.34}$$

$$= \left\langle x_{\text{ZPF}}^2\left(2\hat{n} + 1\right) \right\rangle \tag{8.35}$$

As a result:

$$\big\langle \hat{x}(t)\,\hat{x}(0)\big\rangle = \underbrace{\big\langle \hat{x}(0)\,\hat{x}(0)\big\rangle}_{(8.35)} \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} + \underbrace{\big\langle \hat{p}(0)\,\hat{x}(0)\big\rangle}_{(8.33)} \frac{1}{m\omega_0}\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \tag{8.36}$$

$$= x_{\text{ZPF}}^2 \left[\langle 2\hat{n} + 1\rangle \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} - \langle 1\rangle \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2}\right] \tag{8.37}$$

$$= x_{\text{ZPF}}^2 \left[\langle \hat{n}\rangle\, e^{i\omega_0 t} + \langle \hat{n} + 1\rangle\, e^{-i\omega_0 t}\right] \tag{8.38}$$

The quantum spectral density of the harmonic oscillator is then:

$$\boxed{S_{xx}(\omega) = 2\pi x_{\text{ZPF}}^2 \left[\underbrace{\langle \hat{n}\rangle\ \delta(\omega + \omega_0)}_{\text{oscillator emits energy}} + \underbrace{\langle \hat{n} + 1\rangle\, \delta(\omega - \omega_0)}_{\text{oscillator absorbs energy}}\right]} \tag{8.39}$$

Note: The quantum spectral density is asymmetric! This is because the autocorrelation function is complex, since $\hat{x}$ does not commute with itself at different times.

In the high temperature limit, $k_B T \gg \hbar\omega_0$, we retrieve a symmetric and purely classical spectral density:

$$\langle \hat{n}\rangle \sim \langle \hat{n} + 1\rangle \sim \frac{k_B T}{\hbar\omega_0} \tag{8.40}$$

$$\Longrightarrow \quad \lim_{k_B T \gg \hbar\omega_0} S_{xx}(\omega) = \pi\,\frac{k_B T}{m\omega_0^2}\,\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right] = S_x(\omega) \tag{8.41}$$

In the classical limit, we should retrieve the equipartition theorem:

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\lim_{k_B T \gg \hbar\omega_0} S_{xx}(\omega) \right) \, \mathrm{d}\omega = \frac{1}{2} \frac{k_B T}{m\omega_0^2} \cdot 2 \tag{8.42}$$

$$\langle x^2 \rangle = \frac{k_B T}{m\omega_0^2} \tag{8.43}$$

$$\frac{1}{2} m\omega_0^2 \langle x^2 \rangle = \frac{1}{2} k_B T \qquad \checkmark \tag{8.44}$$

In the low temperature limit, $k_B T \ll \hbar\omega_0$. In that case:

$$\langle \hat{n} \rangle \sim 0 \quad , \quad \langle \hat{n} + 1 \rangle \sim 1 \tag{8.45}$$

$$\implies \quad \lim_{k_B T \ll \hbar\omega_0} S_{xx}(\omega) = 2\pi x_{\text{ZPF}}^2 \delta(\omega - \omega_0) \tag{8.46}$$

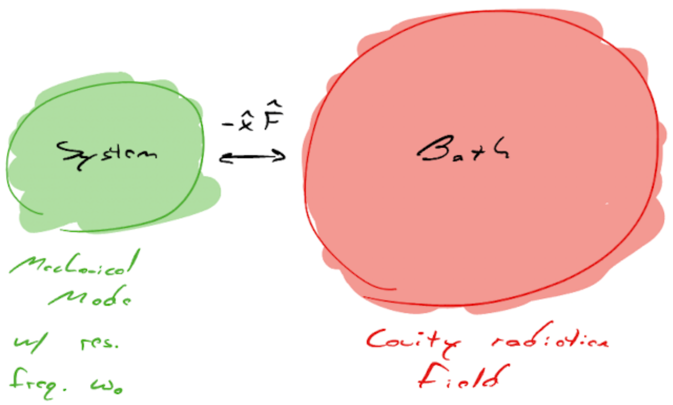
Deep in the quantum regime:

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\lim_{k_B T \ll \hbar\omega_0} S_{xx}(\omega) \right) \, \mathrm{d}\omega = x_{\text{ZPF}}^2 \tag{8.47}$$

$$\langle x^2 \rangle = x_{\text{ZPF}}^2 \qquad \checkmark \tag{8.48}$$

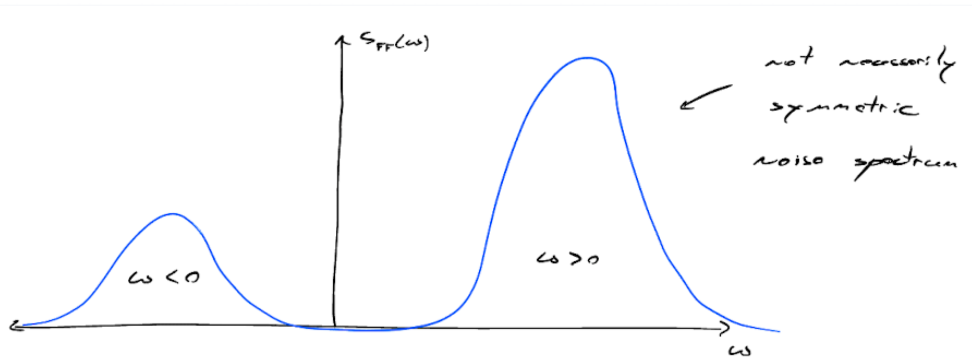
8.3 Quantum Picture Cavity Cooling

Now that we have seen how quantum spectral densities look, let us get back to a quantum treatment of cavity cooling. Let us suppose that our mechanical mode (harmonic oscillator) is coupled to the radiation field of a cavity:



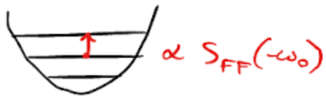
The coupling goes through a radiation pressure force $\hat{F}$, whose spectral density is:

$$S_{FF}(\omega) = \int_{-\infty}^{\infty} \langle \hat{F}(t) \, \hat{F}(0) \rangle \, \mathrm{e}^{i\omega t} \, \mathrm{d}t \tag{8.49}$$



Noise in $\hat{F}$ at the oscillator frequency ω_0 can cause transitions between its harmonic oscillator eigenstates. If $\hat{F}$ is small and the noise has a short correlation time, we can use perturbation theory to derive the transition rates. One finds that the transition rate from $|n\rangle$ to $|n+1\rangle$ is:

$$\Gamma_{n\to n+1} = \frac{x_{\text{ZPF}}^2}{\hbar^2}\,(n+1)\,S_{FF}(-\omega_0) \tag{8.50}$$



and from $|n\rangle$ to $|n-1\rangle$:

$$\Gamma_{n\to n-1} = \frac{x_{\text{ZPF}}^2}{\hbar^2}\,n\,S_{FF}(\omega_0) \tag{8.51}$$

This is according to Fermi's Golden Rule. For further details, see Review of Modern Physics 82, 1155 (2010).

What is S_{FF} from radiation pressure in the optical cavity? Classically, we wrote:

$$\mathscr{F} = \frac{2nh\nu}{c} = \frac{2Nh\nu}{c\Delta t} \qquad n : \text{photons per time, } N/\Delta t \tag{8.52}$$

$$\mathscr{F} = \frac{\hbar\omega_c}{c\frac{\Delta t}{2}}\,N = \frac{\hbar\omega_c}{L}\,N \qquad L : \text{cavity length} \tag{8.53}$$

Quantum:

$$\hat{F} = \left(\frac{\hbar\omega_c}{L}\right)\,\hat{n}_c \qquad \longleftarrow \text{number operator for cavity photons} \tag{8.54}$$

For a fluctuating force $\hat{F}$ acting on a quantum harmonic oscillator, we can say that we have an extra term in the Hamiltonian:

$$\hat{U} = -\hat{x}\hat{F} = -x_{\text{ZPF}}(\hat{a}^+ + \hat{a})\,\hat{F} \tag{8.55}$$

This fluctuating force will cause transitions between oscillator eigenstates. The corresponding Fermi Golden Rule rates come from an application of time-dependent perturbation theory and are related to $S_{FF}(\omega)$. As shown before, the transition rate from oscillator eigenstate $|n\rangle$ to $|n+1\rangle$ is:

$$\Gamma_{n\to n+1} = \frac{x_{\text{ZPF}}^2}{\hbar^2}\,(n+1)\,S_{FF}(-\omega_0) = (n+1)\,\Gamma_{\uparrow} \tag{8.56}$$

and from $|n\rangle$ to $|n-1\rangle$ the rate is:

$$\Gamma_{n\to n-1} = \frac{x_{\text{ZPF}}^2}{\hbar^2}\,n\,S_{FF}(\omega_0) = n\,\Gamma_{\downarrow} \tag{8.57}$$

Given these rates, we can write a master equation for the probability of being in oscillator state $|n\rangle$: $p_n(t)$.

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = \underbrace{\left[n\Gamma_\uparrow p_{n-1} + (n+1)\Gamma_\downarrow p_{n+1}\right]}_{\text{transitions into } |n\rangle} - \underbrace{\left[n\Gamma_\downarrow p_n + (n+1)\Gamma_\uparrow p_n\right]}_{\text{transitions out of } |n\rangle} \tag{8.58}$$

Note that the average energy of the oscillator is:

$$\langle E(t)\rangle = \sum_{\mathrm{n}=0}^{\infty} \hbar\omega_0 \left(n + \frac{1}{2}\right) p_n(t) \tag{8.59}$$

Its time derivative is then:

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \sum_{\mathrm{n}=0}^{\infty} \hbar\omega_0 \left(n + \frac{1}{2}\right) \frac{\mathrm{d}p_n}{\mathrm{d}t} \tag{8.60}$$

$$\begin{aligned}\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} =& \sum_{\mathrm{n}=0}^{\infty} -\hbar\omega_0 \left(n + \frac{1}{2}\right) \left(n\Gamma_\downarrow p_n + (n+1)\Gamma_\uparrow p_n\right) \\ &+ \sum_{m=0}^{\infty} \hbar\omega_0 \left(m + \frac{1}{2} + 1\right) (m+1)\,\Gamma_\uparrow p_m \\ &\quad \text{we replaced } m = n-1 \;\Rightarrow\; n = m+1 \\ &+ \sum_{q=0}^{\infty} \hbar\omega_0 \left(q + \frac{1}{2} - 1\right) q\Gamma_\downarrow p_q \\ &\quad \text{we replaced } q = n+1 \;\Rightarrow\; n = q-1\end{aligned} \tag{8.61}$$

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \sum_{\mathrm{n}=0}^{\infty} \hbar\omega_0 (n+1) p_n \Gamma_\uparrow - \sum_{\mathrm{n}=0}^{\infty} \hbar\omega_0 n p_n \Gamma_\downarrow \tag{8.62}$$

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \underbrace{\sum_{\mathrm{n}=0}^{\infty} \hbar\omega_0 \left(n + \frac{1}{2}\right) p_n}_{\langle E\rangle} \left[\Gamma_\uparrow - \Gamma_\downarrow\right] + \underbrace{\sum_{\mathrm{n}=0}^{\infty} \hbar\omega_0 p_n \frac{1}{2}}_{\frac{\hbar\omega_0}{2}} \left[\Gamma_\uparrow + \Gamma_\downarrow\right] \tag{8.63}$$

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \frac{\hbar\omega_0}{2}\left[\Gamma_\downarrow + \Gamma_\uparrow\right] - \langle E\rangle \left[\Gamma_\downarrow - \Gamma_\uparrow\right] \tag{8.64}$$

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \frac{\hbar\omega_0}{2} \frac{x_{\mathrm{ZPF}}^2}{\hbar^2} \left[S_{FF}(\omega_0) + S_{FF}(-\omega_0)\right] - \langle E\rangle \frac{x_{\mathrm{ZPF}}^2}{\hbar^2} \left[S_{FF}(\omega_0) - S_{FF}(-\omega_0)\right] \tag{8.65}$$

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = \underbrace{\frac{1}{4m} \left[S_{FF}(\omega_0) + S_{FF}(-\omega_0)\right]}_{P} - \frac{\langle E\rangle}{m} \underbrace{\frac{1}{2\hbar\omega_0} \left[S_{FF}(\omega_0) - S_{FF}(-\omega_0)\right]}_{\Gamma} \tag{8.66}$$

$$\frac{\mathrm{d}\langle E\rangle}{\mathrm{d}t} = P - \left(\frac{\Gamma}{m}\right) \langle E\rangle \tag{8.67}$$

We can see that P represents the heating of the oscillator by the noise source, and Γ/m represents the damping of the oscillator by the noise sourve. The heating effect is the result of a random force causing the oscillator's momentum to diffuse, causing $\langle E \rangle$ to grow linearly in time. This is due to the symmetry in the frequency part of the quantum spectral density $(S_{FF}(\omega_0)+S_{FF}(-\omega_0))$. The damping effect is caused by the net tendency of the noise to absorb energy from, rather than emit energy to the oscillator. This is due to the asymmetry in the frequency part of the quantum spectral density $(S_{FF}(\omega_0) - S_{FF}(-\omega_0)$.

This yields the quantum version of the fluctuation dissipation theorem. If the system is in thermal equilibrium, then the transition rates must satisfy the detailed balance relation:

$$\Gamma_{n\to n+1} = \Gamma_{n\to n-1} \tag{8.68}$$

$$(\bar{n}+1)\, S_{FF}(-\omega_0) = \bar{n}\, S_{FF}(\omega_0) \tag{8.69}$$

$$\frac{\bar{n}+1}{\bar{n}} = \frac{S_{FF}(\omega_0)}{S_{FF}(-\omega_0)} \tag{8.70}$$

$$1 + \frac{1}{\bar{n}} = \frac{S_{FF}(\omega_0)}{S_{FF}(-\omega_0)} \tag{8.71}$$

Recall that at thermal equilibrium:

$$\langle E \rangle = \hbar\omega_0 \left(\frac{1}{2} + \underbrace{\frac{1}{\mathrm{e}^{\hbar\omega_0/k_B T} - 1}}_{\bar{n}} \right) = \hbar\omega_0 \left(\bar{n} + \frac{1}{2} \right) \tag{8.72}$$

$$\tag{8.73}$$

Therefore:

$$\mathrm{e}^{\hbar\omega_0/k_B T} = \frac{S_{FF}(\omega_0)}{S_{FF}(-\omega_0)} \tag{8.74}$$

This allows us to relate the symmetric and asymmetric spectral densities:

$$\frac{S_{FF}(\omega_0)+S_{FF}(-\omega_0)}{S_{FF}(\omega_0)-S_{FF}(-\omega_0)} = \frac{1+\mathrm{e}^{-\hbar\omega_0/k_B T}}{1-\mathrm{e}^{-\hbar\omega_0/k_B T}} = \coth\left(\frac{\hbar\omega_0}{2k_B T}\right) \tag{8.75}$$

$$\implies \quad \underbrace{[S_{FF}(\omega_0)+S_{FF}(-\omega_0)]}_{\substack{\bar{S}_{FF}(\omega_0) \\ \text{symmetric quantum} \\ \text{spectral density}}} = \coth\left(\frac{\hbar\omega_0}{2k_B T}\right) \underbrace{[S_{FF}(\omega_0)-S_{FF}(-\omega_0)]}_{\substack{2\hbar\omega_0\Gamma(\omega_0) \\ \text{quantum definition} \\ \text{of dissipation}}} \tag{8.76}$$

The symmetric quantum spectral density $\bar{S}_{FF}(\omega_0)$ is the quantum version of a classical single-sided power spectral density $\bar{S}_f(\omega_0)$. Equation (8.76) can then be expressed as:

$$\boxed{\bar{S}_{FF}(\omega_0) = \coth\left(\frac{\hbar\omega_0}{2k_B T}\right) 2\hbar\omega_0\Gamma(\omega_0)} \tag{8.77}$$

This signifies that noise and dissipation are related by temperature in equilibrium. This is the *quantum version* of the *Fluctuation-Dissipation theorem.*

For high temperatures, $k_B T \gg \hbar\omega_0$, we recover the classical fluctuation-dissipation theorem:

$$\bar{S}_{FF}(\omega_0) = \frac{1+\mathrm{e}^{-\hbar\omega_0/k_B T}}{1-\mathrm{e}^{-\hbar\omega_0/k_B T}}\, 2\hbar\omega_0\Gamma(\omega_0) \tag{8.78}$$

$$\approx \frac{2}{\frac{\cancel{\hbar\omega_0}}{k_B T}}\, 2\,\cancel{\hbar\omega_0}\,\Gamma(\omega_0) \tag{8.79}$$

$$\boxed{\bar{S}_{FF}(\omega_0) \approx 4k_B T\,\Gamma(\omega_0)} \qquad \checkmark \tag{8.80}$$

Note that the quantum version of the Fluctuation-Dissipation theorem can be rewritten in terms of the average occupation number $\bar{n}$, since:

$$\bar{n} = \frac{1}{e^{\hbar\omega_0/k_B T} - 1} \tag{8.81}$$

$$2\bar{n} + 1 = \frac{2 + e^{\hbar\omega_0/k_B T} - 1}{e^{\hbar\omega_0/k_B T} - 1} = \frac{e^{\hbar\omega_0/k_B T} + 1}{e^{\hbar\omega_0/k_B T} - 1} \tag{8.82}$$

$$2\bar{n} + 1 = \coth\left(\frac{\hbar\omega_0}{2k_B T}\right) \tag{8.83}$$

$$\implies \boxed{\bar{S}_{FF}(\omega_0) = 2\hbar\omega_0(2\bar{n}+1)\,\Gamma(\omega_0)} \tag{8.84}$$

This expression also allows us to solve for the average occupation number given a known quantum spectral density of force fluctuations:

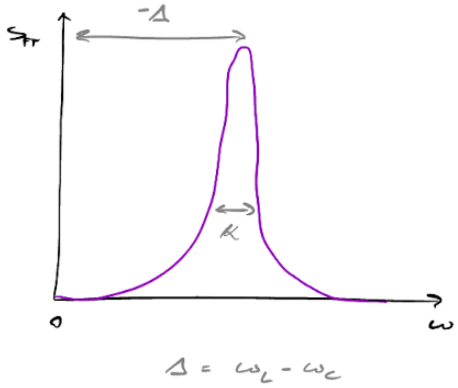
$$2\bar{n} + 1 = \frac{1}{2\hbar\omega_0}\frac{\bar{S}_{FF}(\omega_0)}{\Gamma(\omega_0)} \tag{8.85}$$

$$2\bar{n} + 1 = \frac{S_{FF}(\omega_0) + S_{FF}(-\omega_0)}{S_{FF}(\omega_0) - S_{FF}(-\omega_0)} \tag{8.86}$$

$$\bar{n} = \frac{S_{FF}(-\omega_0)}{S_{FF}(\omega_0) - S_{FF}(-\omega_0)} \tag{8.87}$$

$$\frac{1}{\bar{n}} = \frac{S_{FF}(\omega_0)}{S_{FF}(-\omega_0)} - 1 \tag{8.88}$$

Given a quantum force noise spectral density, we can now see what its effect will be on the mechanical mode's average occupation number $\bar{n}$.



The noise spectral density of this optical force is given by:

$$S_{FF}(\omega) = \left(\frac{\hbar\omega_c}{L}\right)^2 \bar{n}_c \underbrace{\frac{\kappa}{(\omega+\Delta)^2 + \left(\frac{\kappa}{2}\right)^2}}_{\text{photon shot noise spectrum}} \tag{8.89}$$

- κ is the cavity decay rate
- $\Delta = \omega_L - \omega_c$ is the cavity detuning

Assuming that the force noise due to the cavity dominates our other sources (this implies that optical damping dominates over intrinsic damping: $\Gamma_{\text{opt}} \gg \Gamma$), we can use equations (8.88) and (8.89) to solve for the equilibrium occupation number $\bar{n}$:

$$\frac{1}{\bar{n}} = \frac{(-\omega_0+\Delta)^2 - (\omega_0+\Delta)^2}{(\omega_0+\Delta)^2 + \left(\frac{\kappa}{2}\right)^2} \tag{8.90}$$

$$\bar{n} = -\frac{(\omega_0+\Delta)^2 + \left(\frac{\kappa}{2}\right)^2}{4\omega_0\Delta} \tag{8.91}$$

The phonon occupation $\bar{n}$ is minimum for a cavity detuning given by solving $\mathrm{d}\bar{n}/\mathrm{d}\Delta = 0$ for Δ:

$$\Delta_{\min} = -\sqrt{\omega_0^2 + \left(\frac{\kappa}{2}\right)^2} \tag{8.92}$$

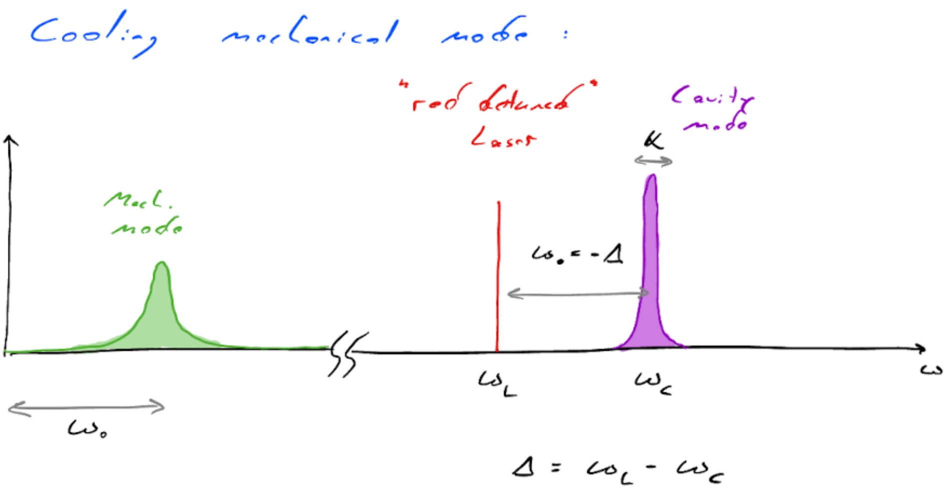
For high-finesse cavities, we have the so-called "resolved sideband" regime: $\omega_0 \gg \kappa$:

$$\boxed{\begin{aligned} &\Delta_{\min} \approx -\omega_0 && (8.93)\\ \implies \quad &\bar{n}_{\min} = \left(\frac{\kappa}{4\omega_0}\right)^2 && (8.94)\end{aligned}}$$

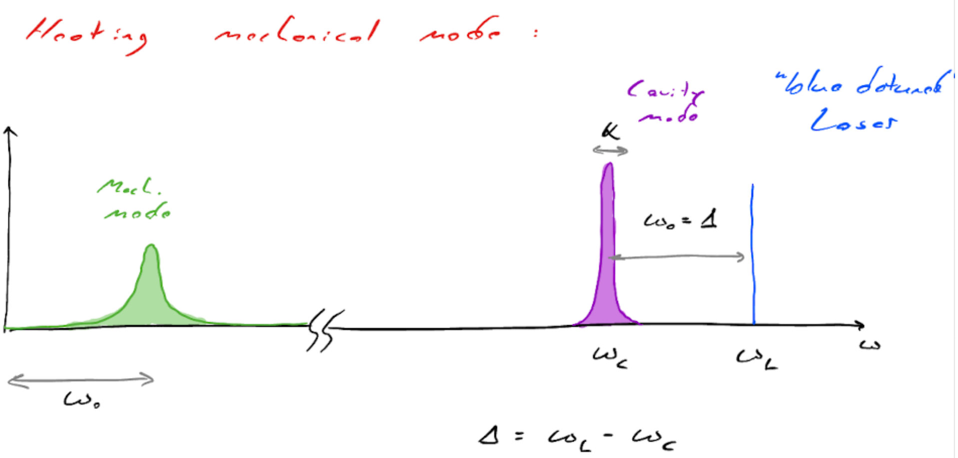
Therefore, to cool the mechanical resonator, the linewidth κ should be small and the mechanical frequency ω_0 should be large.

Graphical Explanation

A laser photon (ω_l) is combines with a phonon from the resonator (ω_0) to produce a cavity photon (ω_c). This extracts phonons from the resonator and releases photons in the cavity:



The reverse can also be done for a detuning of $\Delta = +\omega_0$. Here a laser photon (ω_l) excites a phonon from the resonator (ω_0) and a cavity photon (ω_c):



9 Standard Quantum Limit

9.1 Zero-point motion

In earlier lectures, we have considered a quantum mechanical harmonic oscillator without dissipation. The resulting quantum spectral density of its displacement $x(t)$ was found to be:

$$S_{xx}(\omega) = 2\pi x_{\text{ZPF}}^2 \left[\langle \hat{n} \rangle \; \delta(\omega + \omega_0) + \langle \hat{n} + 1 \rangle \; \delta(\omega - \omega_0)\right] \tag{9.1}$$

$$\text{where } x_{\text{ZPF}}^2 = \frac{\hbar}{2m\omega_0}$$

If we now introduce a mechanical dissipation Γ, we can simply replace the delta functions by properly normalized Lorentzian functions with spectral width proportional to Γ (this approximation is valid for weak Γ). Then:

$$S_{xx}(\omega) = 2\pi x_{\text{ZPF}}^2 \left[\langle \hat{n} \rangle \; \frac{\Gamma}{2\pi m} \frac{1}{(\omega_0 + \omega)^2 + \left(\frac{\Gamma}{2m}\right)^2} + \langle \hat{n} + 1 \rangle \; \underbrace{\frac{\Gamma}{2\pi m} \frac{1}{(\omega_0 - \omega)^2 + \left(\frac{\Gamma}{2m}\right)^2}}_{\substack{L(\omega) \\ \int_{-\infty}^{\infty} L(\omega)\, \mathrm{d}\omega = 1}} \right] \tag{9.2}$$

In the classical high-temperature limit ($k_B T \gg \hbar\omega_0$), we have:

$$\langle \hat{n} \rangle \sim \langle \hat{n} + 1 \rangle \sim \frac{k_B T}{\hbar \omega_0} \tag{9.3}$$

$$\lim_{k_B T \gg \hbar\omega_0} S_{xx}(\omega) = \pi \frac{k_B T}{m\omega_0^2} \frac{\Gamma}{2\pi m} \left[\frac{1}{(\omega_0 + \omega)^2 + \left(\frac{\Gamma}{2m}\right)^2} + \frac{1}{(\omega_0 - \omega)^2 + \left(\frac{\Gamma}{2m}\right)^2} \right] \tag{9.4}$$

$$= S_x(\omega) \qquad \longleftarrow \text{ Classical PSD} \tag{9.5}$$

We should retrieve the equipartition theorem:

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\lim_{k_B T \gg \hbar\omega_0} S_{xx}(\omega) \right) \mathrm{d}\omega = \frac{k_B T}{2m\omega_0^2} \cdot 2 \tag{9.6}$$

$$\langle x^2 \rangle = \frac{k_B T}{m\omega_0^2} \tag{9.7}$$

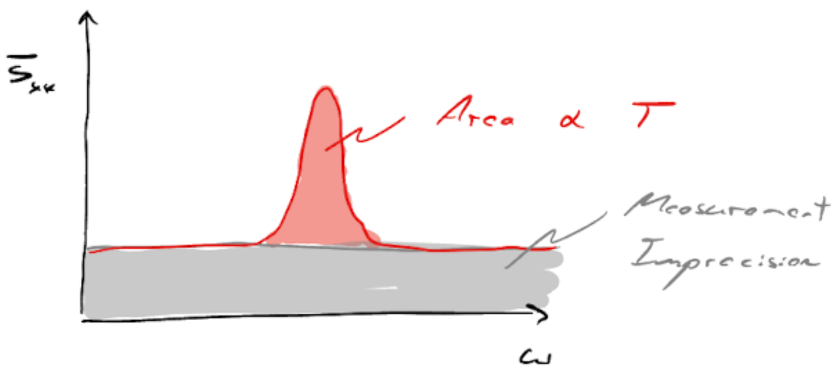
$$\frac{1}{2} m\omega_0 \langle x^2 \rangle = \frac{1}{2} k_B T \qquad \checkmark \tag{9.8}$$

In a measurement, we couple to the symmetric-in-frequency single-sided spectral density:

$$\bar{S}_{xx}(\omega) = S_{xx}(\omega) + S_{xx}(-\omega) \tag{9.9}$$

$$\bar{S}_{xx}(\omega) \approx \frac{\Gamma x_{\text{ZPF}}^2}{m} \left[\langle \hat{n} \rangle + \langle \hat{n} + 1 \rangle \right] \frac{1}{(\omega_0 - |\omega|)^2 + \left(\frac{\Gamma}{2m}\right)^2} \tag{9.10}$$

Classical Limit:



At $T = 0$, in the quantum limit, i.e., $\langle \hat{n} \rangle = 0$, the power spectral density of the resonator motion is given by:

$$\bar{S}^0_{xx}(\omega) = \frac{\Gamma x^2_{\text{ZPF}}}{m} \frac{1}{(\omega_0 - |\omega|)^2 + \left(\frac{\Gamma}{2m}\right)^2} \tag{9.11}$$

$$\bar{S}^0_{xx}(\omega_0) = \frac{4m x^2_{\text{ZPF}}}{\Gamma} = \frac{2\hbar}{\omega_0 \Gamma} \tag{9.12}$$

9.2 Measurement Imprecision

One might therefore expect to measure the spectral density $\bar{S}^0_{xx}(\omega)$ for a nanomechanical resonator cooled to its ground state. However, this ignores both the spectral density of the measurement imprecision (in the case of an interferometric displacement detector, this comes from the shot noise) and the spectral density of the back-action force due to the measurement (in the case of an interferometer, this comes from the photon pressure). Let us now consider the imprecision and the back-action of an optical interferometry measurement. The results from this analysis will be generalized to other types of displacement detectors.

In a cavity displacement detector, changes in the displacement x of the mechanical element leads to changes in phase of the cavity's reflected carrier signal. The changing phase shift is then converted to a changing intensity by the interference effect. A coherent photon state within the cavity contains a Poisson distribution of the number of photons, implying that

$$\text{fluctuations} \longrightarrow \quad (\Delta N)^2 = \bar{N} \quad \longleftarrow \text{mean} \tag{9.13}$$

The uncertainty in any measurement of the phase of this state for large $\bar{N}$ is

$$(\Delta\Theta)^2 = \frac{1}{4\bar{N}} \tag{9.14}$$

For more details, see *Rev. Mod. Phys. 82, 1155 (2010), app. G*. Therefore, large-$\bar{N}$ coherent states obey the number-phase uncertainty relation:

$$\Delta N \, \Delta\Theta = \frac{1}{2} \, , \tag{9.15}$$

which is analogous to the position-momentum uncertainty relation. In terms of spectral densities, this leads to:

$$\underbrace{S_{\dot{N}\dot{N}}}_{\substack{\text{spectral density}\\ \text{of photon flux}}} \cdot \underbrace{S_{\Theta\Theta}}_{\substack{\text{spectral density}\\ \text{of phase fluctuations}}} = \frac{1}{4} \tag{9.16}$$

$$\Longrightarrow \quad \underbrace{\sqrt{S_{\dot{N}\dot{N}}\, S_{\Theta\Theta}} = \frac{1}{2}}_{\substack{\text{wave-particle relation}\\ \text{for coherent beams}}} \tag{9.17}$$

Taking the case of light reflecting off a mirror. The beam will have a phase shift $2k_x$ if the mirror moves by x. Thus, a position uncertainty x is equivalent to phase uncertainty $2k_x$. In spectral densities:

$$\text{imprecision of displace-ment measurement} \longrightarrow \quad S^I_{xx} = \frac{1}{4k^2}\, S_{\Theta\Theta} \quad \longleftarrow \text{corresponding imprecision of phase measurement} \tag{9.18}$$

At the same time, the back-action imported by a photon hitting the mirror corresponds to a momentum $2\hbar k$ per photon. Therefore, the photon shot noise $S_{\dot{N}\dot{N}}$ will correspond to a random back-action force:

$$\text{back action force} \longrightarrow \quad S^I_{FF} = 4\hbar^2 k^2\, S_{\dot{N}\dot{N}} \quad \longleftarrow \text{photon flux spectral density (shot noise)} \tag{9.19}$$

Together, these relations give:

$$S_{FF}\, S^I_{xx} = \hbar^2 S_{\dot{N}\dot{N}}\, S_{\Theta\Theta} = \frac{\hbar^2}{4} \tag{9.20}$$

The quantum limit on the noise of a displacement detector is then:

$$\boxed{\sqrt{S_{FF}\, S^I_{xx}} = \frac{\hbar}{2}} \tag{9.21}$$

This is the *minimum* product of back-action and imprecision for an *ideal* apparatus. In general, the product is larger than this.

9.3 Total Measurement Noise

Now that we understand the quantum limits of displacement measurements, let us go back to the spectral density of our harmonic oscillator at $T = 0$. The spectral density that we will measure is the sum of the zero-point fluctuations *plus* the measurement uncertainty *plus* the displacement fluctuations cased by the measurement back-action:

$$\bar{S}_{xx,\text{tot}} = \underbrace{\bar{S}^0_{xx}(\omega)}_{\substack{\text{zero-point}\\ \text{motion}}} + \underbrace{\bar{S}^I_{xx}(\omega)}_{\substack{\text{measurement}\\ \text{imprecision}}} + \underbrace{|\chi_m(\omega)|^2\, \bar{S}_{FF}(\omega)}_{\text{back-action}} \tag{9.22}$$

$$\bar{S}^I_{xx}(\omega) = S^I_{xx}(\omega) + S^I_{xx}(-\omega) \tag{9.23}$$

$$\bar{S}_{FF}(\omega) = S_{FF}(\omega) + S_{FF}(-\omega) \tag{9.24}$$

$\chi_m(\omega)$ is the mechanical susceptibility of the harmonic oscillator: It gives the displacement in response to a driving force:

$$x(\omega) = \chi_m(\omega)\, F(\omega) \tag{9.25}$$

$$\text{with } \chi_m(\omega) = \frac{1}{m}\, \frac{1}{\omega_0^2 - \omega^2 + i\frac{\Gamma\omega}{m}}$$

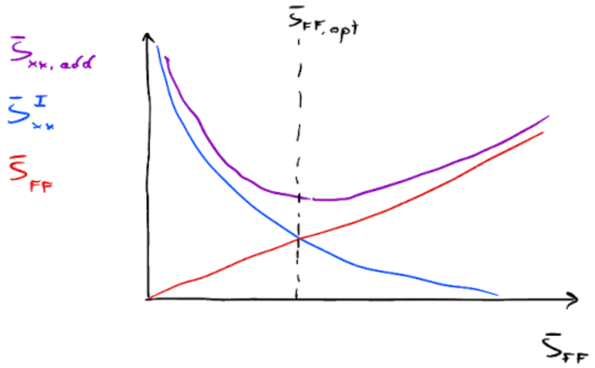
Let us assume that we have a quantum-limited (best possible) detector, i.e., with $S_{FF} \cdot S^I_{xx} = \frac{\hbar^2}{4}$. If the shot noise spectral density of our detector is symmetric in frequency, then

$$\bar{S}^I_{xx}(\omega) = 2S^I_{xx}(\omega) \tag{9.26}$$

$$\bar{S}_{FF}(\omega) = 2S_{FF}(\omega) \tag{9.27}$$

$$\longrightarrow \quad \bar{S}^I_{xx} = \frac{\hbar^2}{\bar{S}_{FF}(\omega)} \tag{9.28}$$

$$\Longrightarrow \quad \bar{S}_{xx,\mathrm{tot}}(\omega) = \bar{S}^0_{xx}(\omega) + \underbrace{\frac{\hbar^2}{\bar{S}_{FF}(\omega)} + |\chi_m(\omega)|^2\,\bar{S}_{FF}(\omega)}_{\substack{\bar{S}_{xx,\mathrm{add}}(\omega) \\ \text{this is the additional} \\ \text{position-uncertainty noise}}} \tag{9.29}$$



For high back-action (more photons) imprecision is low, while for low back-action (fewer photons), imprecision is high.

The minimum added noise $\bar{S}_{xx,\mathrm{add}}$ is obtained for an optimal back-action $\bar{S}_{FF,\mathrm{opt}}$, which corresponds to a particular photon intensity (laser power):

$$\bar{S}_{FF,\mathrm{opt}}(\omega) = \frac{\hbar}{|\chi_m(\omega)|} \tag{9.30}$$

If we tune our cavity for minimum additional noise at the resonator resonance frequency ω_0, the we have:

$$\bar{S}_{FF,\mathrm{opt}}(\omega_0) = \frac{\hbar}{|\chi_m(\omega_0)|} = \hbar\omega_0\Gamma \tag{9.31}$$

For this optimal force, the additional noise is:

$$\bar{S}_{xx,\mathrm{add}}(\omega) = \hbar\left(|\chi_m(\omega_0) + \frac{|\chi_m(\omega)|^2}{|\chi_m(\omega_0)|}\right) \tag{9.32}$$

So the total noise is then:

$$\bar{S}_{xx,\mathrm{tot}}(\omega) = \bar{S}^0_{xx}(\omega) \;+\; \hbar\left(|\chi_m(\omega)| \;+\; \frac{|\chi_m(\omega)|^2}{|\chi_m(\omega_0)|}\right) \tag{9.33}$$

Exactly on resonance, this results in:

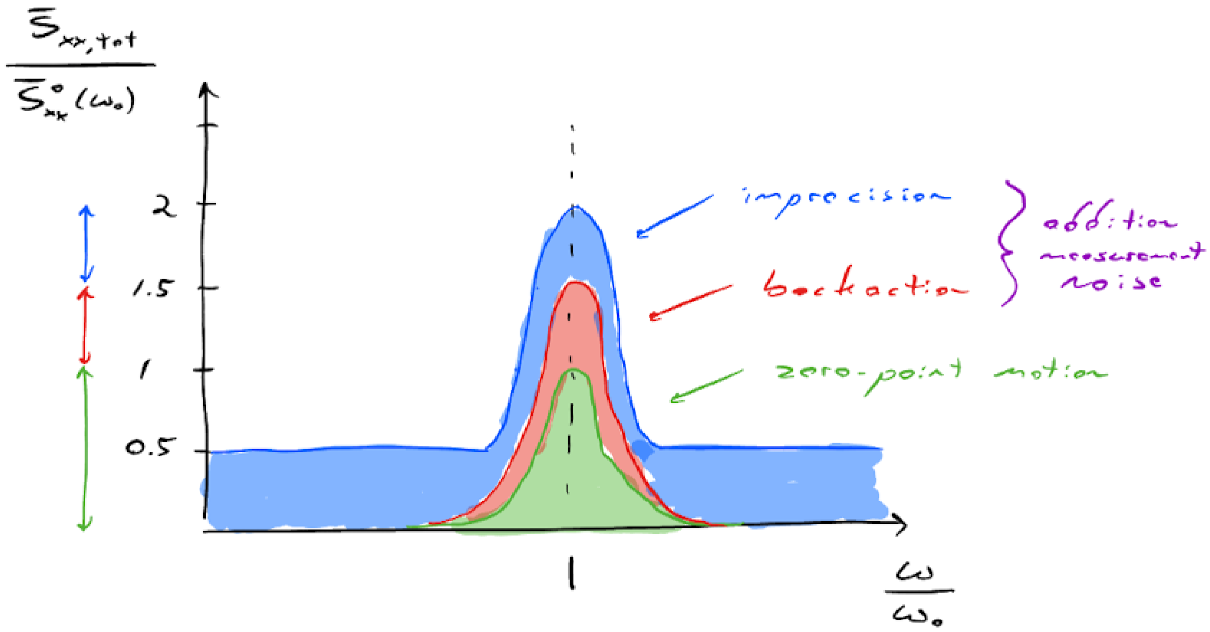
$$\bar{S}_{xx,\text{tot}}(\omega_0) = \bar{S}^0_{xx}(\omega_0) + \hbar|\chi_m(\omega_0)| + \hbar|\chi_m(\omega_0)| \tag{9.34}$$

$$= \frac{2\hbar}{\omega_0\Gamma} + \frac{\hbar}{\omega_0\Gamma} + \frac{\hbar}{\omega_0\Gamma} \tag{9.35}$$

$$= \bar{S}^0_{xx}(\omega_0)\left[1 + \frac{1}{2} + \frac{1}{2}\right] \tag{9.36}$$

$$= 2\bar{S}^0_{xx}(\omega_0) = \frac{4\hbar}{\omega_0\Gamma} \tag{9.37}$$

We see that half of the measured noise is from the resonator itself, and half is from the added noise of the measurement. *This is known as the standard quantum limit on position detection.* Therefore, to reach the quantum limit, one must cool the resonator into the ground state *and* use a quantum limited detector.



10 Nanowires

10.1 Why Nanowires

Nanowires are heavily investigated objects for several reasons. First of all, they can be fabricated in a very controlled manner, from all kinds of materials. Second, they are excellent mechanical oscillators with low-dissipation (Γ) and high resonance frequency (ω_0). The second point we can show by proportionality analyses of a beam with length l, width w, thickness t, and quality factor Q:

$$m \propto w\,t\,l \qquad \omega_0 \propto t/l^2 \qquad Q \propto t \quad \text{known from experience} \tag{10.1}$$

$$\Gamma = \frac{m\omega_0}{Q} \propto \frac{wt}{l} \tag{10.2}$$

If we scale all dimensions l, w, t by a factor β:

$$\Gamma \propto \beta \qquad \omega_0 \propto \beta^{-1} \tag{10.3}$$

We see, that smaller objects have lower dissipation Γ and higher resonance frequencies ω_0. Similarly, the limits of various measurements scales in different order with β:

$$\textbf{Amplitude Measurements} \tag{10.4}$$

$$F_{\text{min}} = \sqrt{4k_BT\Gamma} \propto \sqrt{\frac{wt}{l}} \propto \beta^{1/2} \tag{10.5}$$

$$\tau_{\text{min}} = l_e\sqrt{4k_BT\Gamma} \propto \sqrt{wtl} \propto \beta^{3/2} \tag{10.6}$$

$$\textbf{Frequency Measurements} \tag{10.7}$$

$$\left(\frac{\partial F}{\partial x}\right)_{\text{min}} = \frac{1}{x_{\text{osc}}}\sqrt{4k_BT\Gamma} \propto \frac{wt^2}{l^2} \propto \beta \tag{10.8}$$

$$\left(\frac{\partial \tau}{\partial \Theta}\right)_{\text{min}} = \frac{l_e}{\Theta_{\text{osc}}}\sqrt{4k_BT\Gamma} \propto wt^2 \propto \beta^2 \tag{10.9}$$

Note however:

$$x_{\text{th}} = \sqrt{\frac{k_BT}{m\omega_0^2}} \propto \sqrt{\frac{l^3}{wt^3}} \propto \beta^{-1/2} \tag{10.10}$$

10.2 Duffing Equation

The typical simple equation of motion of a harmonic oscillator is only valid for very small deflections of a nanowire. A better model to describe the motion mathematically is the *Duffing Oscillator*. Its equation of motion has the following form (mind the division of both sides of the equation by m):

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0 x(t) + \alpha x^3(t) = F(t) \tag{10.11}$$

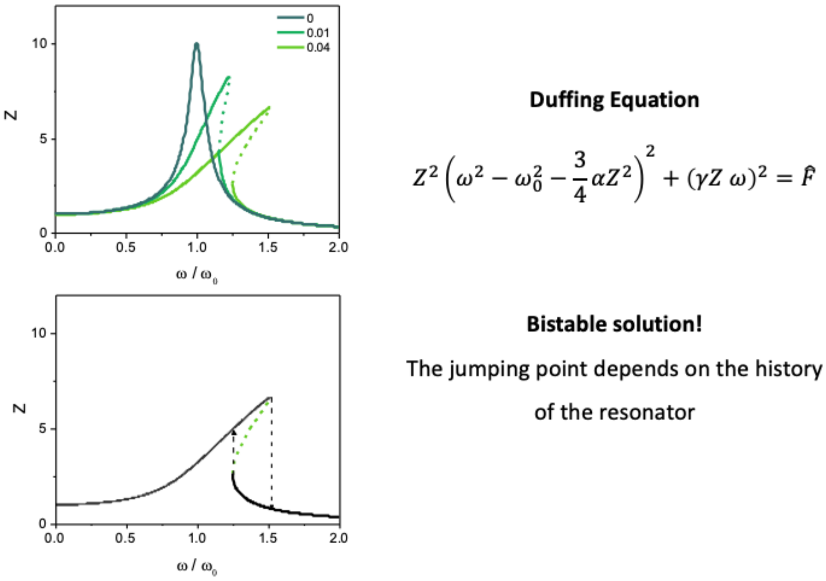
Rearranging yields:

$$\ddot{x}(t) + \gamma\dot{x}(t) + x(t)\left(\omega_0 + \alpha x^2(t)\right) = F(t) \tag{10.12}$$

A positive value of α can be seen as a hardening of the spring constant, and a negative value as a softening of the spring constant. The general solution to this equation is given by

$$x(t) = Z\cos(\omega t - \psi) \tag{10.13}$$

An interesting feature of this system is the occurrence of a hysteresis:



10.3 Two-Mode Equation of Motion

Let us return to the more basic description of a nanowire as a simple harmonic oscillator. Generally, the equation of motion for a harmonic oscillator is given by:

$$m\ddot{x} + \Gamma\dot{x} + kx = F_{\text{th}} + F_0 \tag{10.14}$$

The nanowire has two degrees of freedom for oscillations. Therefore, there are two equations of motion:

$$m\ddot{r}_1 + \Gamma_1\dot{r}_1 + k_1 r_1 = F_{\text{th},1} + F_{0,1} \tag{10.15}$$

$$m\ddot{r}_2 + \Gamma_2\dot{r}_2 + k_2 r_2 = F_{\text{th},2} + F_{0,2} \tag{10.16}$$

This can be generalized to

$$m\ddot{r}_i + \Gamma_i\dot{r}_i + k_i r_i = F_{\text{th}} + F_i \tag{10.17}$$

$$\text{where} \quad F_i = F_{0,i} + \frac{\partial F_i}{\partial r_i} r_i + \frac{\partial F_i}{\partial r_j} r_j + ... \tag{10.18}$$

Keeping only terms up to first order of F_i, we obtain:

$$m\ddot{r}_i + \Gamma_i\dot{r}_i + k_i r_i = F_{\text{th}} + F_{0,i} + \frac{\partial F_i}{\partial r_i} r_i + \frac{\partial F_i}{\partial r_j} r_j \qquad \text{for} \quad {\scriptstyle i=1,2 \atop i\neq j} \tag{10.19}$$

This can be expressed using vectors and matrices:

$$m\ddot{\vec{r}} + \tilde{\Gamma}\,\dot{\vec{r}} + \tilde{k}\,\vec{r} = \vec{F}_{\text{th}} + \vec{F}_0 \ , \tag{10.20}$$

$$\tilde{\Gamma} = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix} \tag{10.21}$$

$$\tilde{k} = \begin{pmatrix} k_1 - \frac{\partial F_1}{\partial r_1} & -\frac{\partial F_2}{\partial r_1} \\ -\frac{\partial F_1}{\partial r_2} & k_2 - \frac{\partial F_2}{\partial r_2} \end{pmatrix} \tag{10.22}$$

11 Dissipation and Quality Factor

11.1 Dissipation

Mechanical dissipation (or the friction coefficient) quantifies the energy loss to the environment by an oscillator. Recall our harmonic oscillator equation of motion:

$$m\ddot{x} + \Gamma\dot{x} + kx = F(t)\ , \qquad \text{with} \quad \underbrace{\Gamma = \frac{m\omega_0}{Q}\left[\frac{kg}{s}\right]}_{\substack{\text{typical definition}\\ \text{in force microscopy}}}\ , \qquad k = m\omega_0^2 \tag{11.1}$$

Equivalent to:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{1}{m}F(t)\ , \qquad \text{with} \quad \underbrace{\gamma = \frac{\omega_0}{Q}\ \ [\text{Hz}]}_{\substack{\text{typical definition}\\ \text{in optomechanics}}} \tag{11.2}$$

The energy lost from time $t = 0$ to t is given by:

$$\Delta E = \int_{x(0)}^{x(t)} (-\Gamma\dot{x})\ \mathrm{d}x = \int_0^t (-\Gamma\dot{x})\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\ \mathrm{d}t \tag{11.3}$$

$$\Delta E = \int_0^t \underbrace{\left(-\Gamma\,\dot{x}^2\right)}_{\substack{\text{rate of}\\ \text{energy loss}}}\ \mathrm{d}t \tag{11.4}$$

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\Gamma\dot{x}^2} \tag{11.5}$$

The energy of the harmonic oscillator is:

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{11.6}$$

Using the solution to the harmonic oscillator equation of motion $x = x_0\,\mathrm{e}^{i\omega t}$, where $\omega = \sqrt{\frac{k}{m}} + i\frac{\Gamma}{2m} = \omega_0 + i\frac{\Gamma}{2m}$ and the system dissipation is low $\frac{\Gamma}{2m} \ll \omega_0$:

$$E(t) = E_0\,\mathrm{e}^{-\Gamma t/m} \qquad \text{with } E_0 = \frac{1}{2}kx_0^2 \tag{11.7}$$

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\Gamma}{m}E} \qquad \text{Recall the same formula as in the quantum definition of dissipation} \tag{11.8}$$

This formula also allows us to understand the quality factor Q. Q is, in fact, sometimes defined as:

$$Q = 2\pi\,\frac{\text{Energy stored in oscillator}}{\text{Energy dissipated in one period}} \tag{11.9}$$

$$Q = 2\pi\,\frac{\cancel{E}}{\left(\frac{\Gamma}{m}\cancel{E}\right)\,T} \qquad T = \frac{2\pi}{\omega_0} \tag{11.10}$$

$$\boxed{Q = \frac{m\omega_0}{\Gamma}} \qquad \text{As before: } \Gamma = \frac{m\omega_0}{Q} \tag{11.11}$$

11.2 Sources of Dissipation

Now that we have a clear idea of what Γ and Q represent, we should understand their origin in real nanomechanical resonators. As we already discussed, these terms determine the fundamental sensitivity limits of nanomechanical sensors.

The total dissipation in a resonator is the sum of dissipation from various sources:

$$\Gamma = \underbrace{\Gamma_{\text{medium}}}_{\substack{\text{losses due to}\\\text{interactions with}\\\text{fluid or ballistic}\\\text{medium}}} + \underbrace{\Gamma_{\text{clamping}}}_{\substack{\text{energy radiation}\\\text{into environment}\\\text{through clamping}\\\text{point}}} + \underbrace{\Gamma_{\text{intrinsic}}}_{\substack{\text{losses within}\\\text{the resonator}\\\text{itself}}} + \underbrace{\Gamma_{\text{other}}}_{\text{the rest}} \tag{11.12}$$

Equivalently, we can write this in terms of Q:

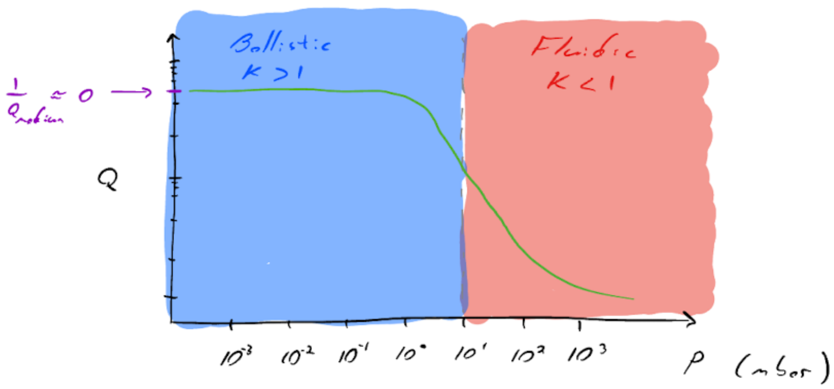
$$\frac{1}{Q} = \frac{1}{Q_{\text{medium}}} + \frac{1}{Q_{\text{clamping}}} + \frac{1}{Q_{\text{intrinsic}}} + \frac{1}{Q_{\text{other}}} \tag{11.13}$$

Medium Dissipation

This dissipation depends on the medium surrounding the resonator, i.e., either a liquid or gas. The highest sensitivity applications are carried out in high vacuum, in which $\Gamma_{\text{medium}} = 0$. In general, in a gas, there are two regimes: the fluidic regime and the ballistic regime. The value of the Knudsen number sets which regime applies to the system:

$$K = \frac{\lambda_f}{L}\ , \tag{11.14}$$

where λ_f is the mean free path of the gas, and L is the representative physical length scale of the resonator. If $K < 1$ ($\lambda_f < L$), the system is in the fluidic regime. If $K > 1$ ($\lambda_f > L$), then the system is in the ballistic regime. In air at atmospheric pressure, $k = 1$ for $L \approx 70$ nm.

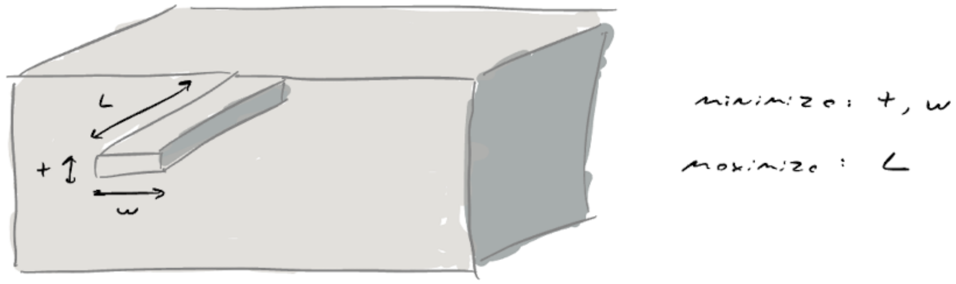


This line can be shifted to higher pressures for smaller resonators.

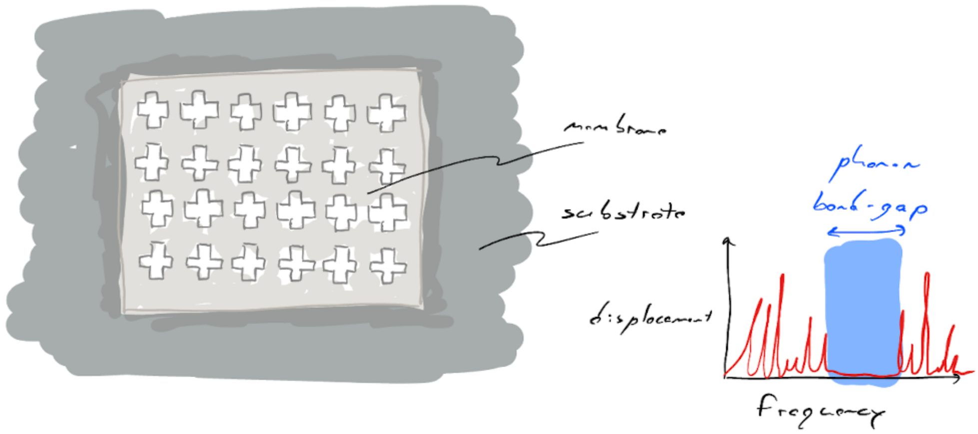
Therefore, to *reduce* Γ_{medium}, one should *minimize the pressure* and the *size of the resonator.*

Clamping Dissipation

This mechanism descries losses due to the radiation of vibrational energy through the resonators anchoring point to the environment. In general, the clamping points need to be designed in order to minimize such radiative losses, if this term is to be made negligible. In the specific case of cantilever beams, Γ_{clamping} is minimized for long and thin beams clamped to a semi-infinite body.



For membranes, a successful way to suppress radiation losses is to locate the vibrating structure within an appropriately designed phononic bandgap structure. This will remove free frame modes around the membrane, suppressing radiation loss. For a recent publication, see Yu, P-L., et al. "A phononic bandgap shield for high-Q membrane microresonators." *Applied Physics Letters* 104.2 (2014): 023510.



Intrinsic Dissipation
This category includes all energy losses that take place within or on the mechanical resonator. These losses can be divided into frictional losses from material imperfections in the bulk or on the surface, and fundamental losses, occurring even in an ideal frictionless material due to phonons and electrons.

Frictional losses come from irreversible motion of atoms during vibration, e.g.g, from defect dislocations in a crystal, grain boundary slipping in a metal, phase boundary slipping in layered structures, or molecular chain motion in an amorphous solid. Such losses can be described by Zener's model for an anelastic solid, which we discussed in chapter 3.2. There we introduced the possibility of time lag in the stress-strain relationship. Ultimately, this resulted in a frequency dependent Q:

$$\frac{1}{Q_{\text{friction}}} = \frac{\omega T}{1+\omega^2 T^2}\,\Delta \tag{11.15}$$

$$\text{with} \quad T = \sqrt{T_\sigma T_\Sigma} \quad \text{and} \quad \Delta = \frac{T_\sigma - T_\Sigma}{T} \ , \tag{11.16}$$

where T_σ and T_Σ are the relaxation times at constant stress and strain, respectively. This results in a frictional dissipation (proportional to $1/Q_{\text{friction}}$) peaked at $\omega = 1/T$.

Fundamental losses include thermoelastic loss and phonon-phonon interaction loss. Like frictional loss, both can be modelled by Zener's approach and therefore have a similar dependence on ω.

- Thermoelastic loss: Mechanical motion generates local differences in temperature and therefore heat flow between these points. This process results in dissipation of energy.
- Phonon-phonon loss: Oscillating strain field changes normal mode frequencies of atomic vibrations in a crystal. Temperature differences in these normal modes result in heat flow and energy loss.

Other Dissipation
This term includes all remaining losses, including dissipation due to electrical charges trapped on the resonator, eddy current damping, magnetic dissipation due to magnetic impurities, etc.

11.3 Damping Dilution

In recent years, a scheme for achieving very high Q resonators has been developed for strings and membranes. In a cantilever, the energy stored and lost over an oscillation cycle is related to its bending. For strings and membranes, energy can also be stored and lost in lateral elongation. Also, strings and membranes can build up a lot of potential energy when vibrational deflection works against high lateral tensile stress.

$$Q = 2\pi \frac{\overbrace{W_{\text{tension}} + W_{\text{elongation}} + W_{\text{bending}}}^{\text{stored energy per cycle}}}{\underbrace{\Delta W_{\text{elongation}} + \Delta W_{\text{bending}}}_{\text{lost energy per cycle}}} \tag{11.17}$$

If we increase the tension on the object ($\rightarrow W_{\text{tension}}$), we increase the stored energy without increasing loss.

Let $\quad Q_{\text{intrinsic}} = 2\pi \dfrac{W_e + W_b}{\Delta W_e + \Delta W_b}$, then:

$$Q = \frac{W_t + W_e + W_b}{W_e + W_b} \cdot Q_{\text{intrinsic}} \tag{11.18}$$

If we increase the tension such that $W_t \gg W_e, W_b$:

$$Q \approx \frac{W_t}{W_e + W_b} \cdot Q_{\text{intrinsic}} \tag{11.19}$$

$$\boxed{\begin{aligned} Q &\approx \alpha_{dd} \cdot Q_{\text{intrinsic}} \\ \text{where} \quad \alpha_{dd} &= \underbrace{\left[\frac{W_e}{W_t} + \frac{W_b}{W_t}\right]^{-1}}_{\text{Damping Dilution factor}} \end{aligned}} \tag{11.20}$$

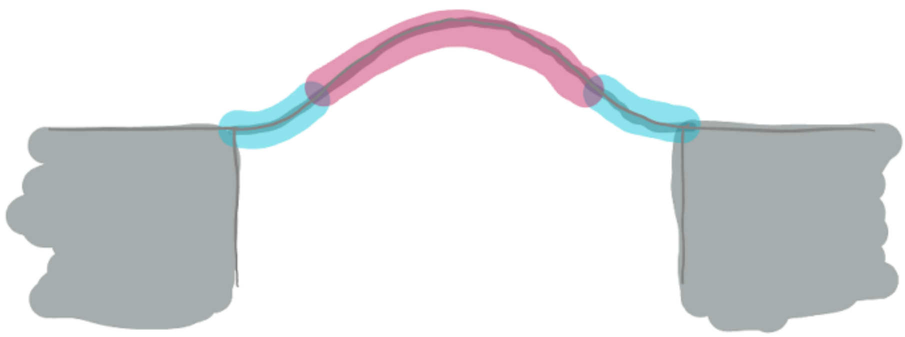
For both strings and membranes, one can calculate the energy stored in elongation, bending, and tension from the mode shapes (see Fund. Nanomech. Resonators, p.81-87). In both cases $\frac{W_b}{W_t}$ is independent of the vibration amplitude, while $\frac{W_e}{W_t}$ depends on the square of the ratio between the vibrational amplitude and the resonator thickness. Since in most cases, the amplitude is much smaller than the thickness, we have:

$$\frac{W_b}{W_t} \gg \frac{W_e}{W_t} \qquad \Longrightarrow \qquad \alpha_{dd} \approx \left[\frac{W_b}{W_t}\right]^{-1} \tag{11.21}$$

The ration $\frac{W_b}{W_t}$ can be separated into two terms:

$$\alpha_{dd} \approx \left[\underbrace{c_1 \left(\frac{h}{L}\right)^2}_{\substack{\text{bending} \\ \text{at center}}} + \underbrace{c_2 \left(\frac{h}{L}\right)}_{\substack{\text{bending} \\ \text{at edge}}} \right]^{-1} \tag{11.22}$$

c_1 and c_2 are constants depending on the mode number, geometry, and material. h is the thickness of the string or membrane and L is its lateral dimension (length for a string; length and width for a membrane).



Given that $\frac{h}{L} \ll 1$, the bending at the edge dominates this effect. Therefore, in order to obtain the highest possible Q, one should apply high tension and minimize bending of the resonator near the clamping points, i.e., realize "soft clamping".

Note: damping dilution increases Q which is useful for frequency standards and optomechanics. It does *not* decrease Γ! Recall that $\Gamma = \frac{m\omega_0}{Q}$. Increasing tension, increases stored energy and also the resonant frequency: $\omega_0 \propto W_t$. Damping dilution gibes $Q \propto W_t$, as well.

$$\implies \quad \Gamma = \frac{m\omega_0}{Q} \propto \frac{W_t}{W_t} = \text{const.} \tag{11.23}$$

Therefore, for example, force and torque sensitivity cannot be improved by damping dilution.

12 Revision

This chapter gives a brief overview over all the main topics discussed in this lecture. It also summarizes the contents and formulas which should be known for the exam (!).

We encountered:

- General principles of stress-strain
- Force & torque balance
- Application of boundary conditions
- Flexural vibrations (of a cantilever)
- Zener's model for an anelastic solid
- Correspondence between beam dynamics and the harmonic oscillator:

  Beam

$$\underbrace{\hat{x}_{\text{end}}}_{\substack{\text{motion at end}\\ \text{of cantilever}}} = \frac{4\overbrace{\hat{F}_p(\omega)}^{\substack{\text{force at end}\\ \text{of cantilever}}}}{m} \frac{1}{\omega_i^2 - \omega^2 + i\frac{\omega_i^2}{Q}} \tag{12.1}$$

$$\hat{x}_{\text{end}}(\omega) = \hat{F}_p(\omega)\,\chi_m(\omega) \tag{12.2}$$

  Harmonic Oscillator

$$\hat{x}(\omega) = \frac{\hat{F}(\omega)}{m} \frac{1}{\omega_0^2 - \omega^2 + i\frac{\omega_0\omega}{Q}} \tag{12.3}$$

$$\hat{x}(\omega) = \hat{F}(\omega)\,\chi_m(\omega) \tag{12.4}$$

- Static & dynamic spring constant:

$$k_D \approx \frac{i}{Q} k_S \tag{12.5}$$

- Concept of power spectral density (PSD):

$$\underbrace{S_x(\omega)}_{\substack{\text{Power Spectral}\\ \text{Density (PSD)}}} = \int_{-\infty}^{\infty} \underbrace{\langle x(t)\,x(0)\rangle}_{\substack{\text{correlation}\\ \text{function}}} \mathrm{e}^{i\omega t}\mathrm{d}t \tag{12.6}$$

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\,\mathrm{d}\omega \tag{12.7}$$

$$S_x(\omega) = \frac{S_F(\omega)}{m^2} \left(\frac{1}{(\omega_0^2 - \omega^2)^2} + \frac{\omega_0^2\omega^2}{Q^2} \right) \tag{12.8}$$

- Fluctuation-dissipation theorem and the limits of measurements:

$$\bar{S}_f(\omega) = 4k_B T\Gamma \tag{12.9}$$

$$\longrightarrow \quad F_{\min} = \sqrt{4k_B T\Gamma} \qquad \left[\frac{N}{\sqrt{Hz}}\right] \tag{12.10}$$

$$\longrightarrow \quad m_{\min} = \frac{2}{x_0\omega_0^2}\sqrt{2k_B T\Gamma} \qquad \left[\frac{kg}{\sqrt{Hz}}\right] \tag{12.11}$$

$$\text{From} \quad \frac{\delta\nu_0}{\nu} = -\frac{1}{2}\frac{\delta m}{m}$$

$$\longrightarrow \quad k_{\min} = \frac{2}{x_0}\sqrt{2k_B T\Gamma} \qquad \left[\frac{N}{m\sqrt{Hz}}\right] \tag{12.12}$$

$$\text{From} \quad \frac{\delta\nu_0}{\nu} = \frac{1}{2}\frac{\delta k}{k}$$

- Transducers (typically nanomechanical elements) and Detectors (many different kinds for displacement)

- Motivations for ground-state cooling ($h\omega_0 \gg k_B T$):
  - ultimate force resolution
  - quantum regime for "macroscopic" objects
  - Measure mechanical superpositions and coherences

- Methods for ground-state cooling:
  - "Brute force"
  - Damping (feedback cooling)
  - Cavity cooling

- Standard Quantum Limit:

$$\Delta x_{\mathrm{SQL}} = x_{\mathrm{ZPF}} = \sqrt{\langle x^2\rangle_0} = \sqrt{\frac{\hbar}{2m\omega_0}} \tag{12.13}$$

- How to achieve cooling by *damping*?

$$N_{\text{mode, min}} = \frac{1}{\hbar}\sqrt{\underbrace{\Gamma}_{\text{reduce dissipation}}\;\underbrace{k_B T}_{\text{reduce starting temperature}}\;\underbrace{\bar{S}_{x_n}}_{\text{reduce detector noise}}} = \frac{1}{2\hbar}\sqrt{\bar{S}_F\,\bar{S}_{x_n}} \tag{12.14}$$

- Cavity cooling & Quantum treatment of fluctuation and dissipation: quantum power spectral density:

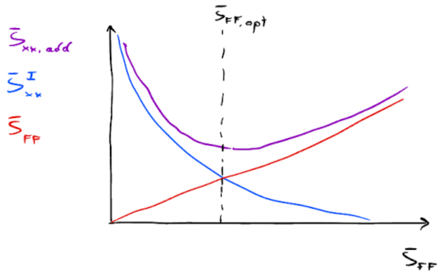
$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \langle \hat{x}(t)\,\hat{x}(0)\rangle\, \mathrm{e}^{i\omega t}\,\mathrm{d}t \tag{12.15}$$

- Quantum limit on the noise of a displacement detector:

$$\frac{\hbar}{2} = \sqrt{S_{FF}\,S^I_{xx}} \tag{12.16}$$

- Total measurement noise:

$$\bar{S}_{xx,\text{tot}} = \underbrace{\bar{S}^0_{xx}(\omega)}_{\text{zero-point motion}} + \overbrace{\underbrace{\bar{S}^I_{xx}(\omega)}_{\text{measurement imprecision}} + \underbrace{|\chi_m(\omega)|^2\,\bar{S}_{FF}(\omega)}_{\text{back-action}}}^{\bar{S}_{xx,\text{add}}} \quad (12.17)$$

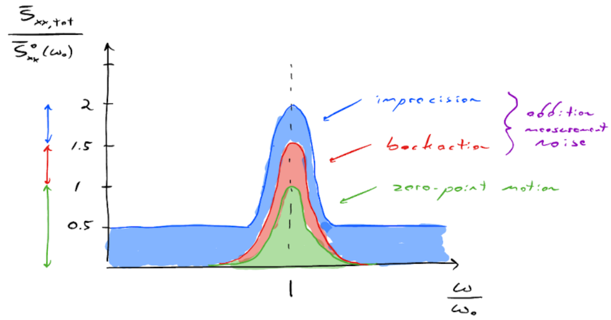


On resonance:

$$\bar{S}_{xx,\text{tot}}(\omega_0) = \bar{S}^0_{xx}(\omega_0)\left[1 + \frac{1}{2} + \frac{1}{2}\right] \quad (12.18)$$

$$= 2\bar{S}^0_{xx}(\omega_0) \quad (12.19)$$

$$= \frac{4\hbar}{\omega_0\Gamma} \quad (12.20)$$



- Mechanical dissipation:

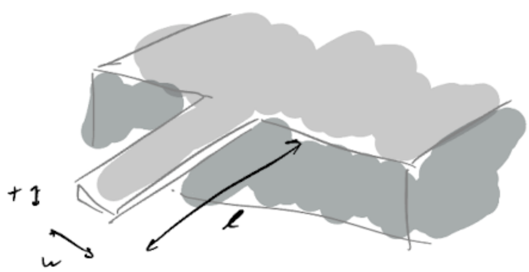
$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\Gamma}{m}E \quad (12.21)$$

$$\Gamma = \frac{m\omega_0}{Q} \quad (12.22)$$

$$\text{Definition of } Q \quad \longrightarrow \quad Q = 2\pi\,\frac{\text{Energy stored}}{\text{Energy dissipated in one period}} \quad (12.23)$$

- Concept of damping dilution: increase energy stored, keep energy loss per cycle the same

- Why nano? For small $F_{\min}$, $k_{\min}$, we need small $\Gamma = \frac{m\omega_0}{Q}$.



$$\text{Beam:} \quad \omega_o \propto \frac{t}{l^2} \quad (12.24)$$

$$Q \propto t \quad \text{experimentally observed} \quad (12.25)$$

$$m \propto wtl \quad (12.26)$$

$$\Longrightarrow \Gamma \propto \frac{wtl \cdot \frac{t}{l^2}}{t} \quad (12.27)$$

$$\underbrace{\Gamma = \frac{wt}{l}}_{\text{lowest dissipation for long and thin cantilevers}} \quad (12.28)$$

Further, if we scale uniformly each dimension by β:

$$\Gamma \propto \beta \quad \longleftarrow \quad \text{smaller, less energy loss} \quad (12.29)$$